MAA OMWATI DEGREECOLLEGE

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NOTES

CLASS:- B.SC 1ST SEM

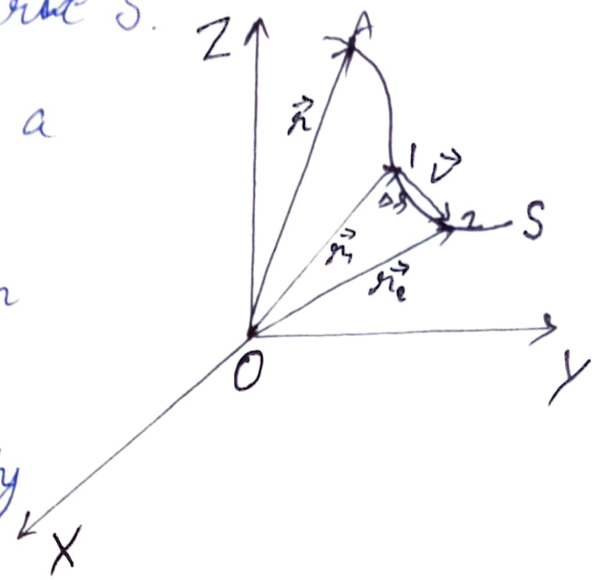
SUBJECT: PHYSICS

(MECHANICS AND THEORY OF RELATIVITY)

(MC)

Mechanics of a particle

Consider a particle of mass 'm' moving along the curve 'S'. Let the particle be at a point A whose position vector from the origin O be $\vec{r}$. Then its instantaneous linear velocity $\vec{v}$, defined as



$$\vec{v} = \frac{d\vec{r}}{dt} \quad \text{——— (1)}$$

If the displacement of particle from point 1 to point 2 is $\Delta\vec{s}$ then

$$\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} \quad \text{——— (2)}$$

Let r_1 and r_2 be the radius vectors of point 1 to 2, then displacement $\Delta\vec{s}$ is given by

$$\Delta s = \vec{r_2} - \vec{r_1} = \Delta\vec{r} \quad \text{——— (3)}$$

Using eq^n (1), (2) and (3), we have

$$\vec{v} = \frac{d\vec{r}}{dt} = \lim_{\Delta t \to 0} \frac{\Delta\vec{s}}{\Delta t} = \frac{d\vec{s}}{dt} \quad \text{——— (4)}$$

$$\vec{v} = \frac{d\vec{s}}{dt}$$

Acc. to Newton's II^{nd} law of motion

$$\vec{F} = \frac{d\vec{p}}{dt} = \frac{d}{dt}(m\vec{v}) \quad \text{——— (5)}$$

mass (m) remains cons. then eqⁿ (5)

$$\vec{F} = m\frac{d\vec{v}}{dt} \quad \text{——— (6)}$$

$\frac{d\vec{v}}{dt}$ is the rate of change of velocity $\vec{v}$ and by def. denotes the acc. $\vec{a}$ of the particle ; so

$$\vec{a} = \frac{d\vec{v}}{dt}$$

$$\vec{a} = \frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right) = \frac{d^2r}{dt^2} \quad \text{——— (7)}$$

Therefore, eqⁿ (5) takes the form

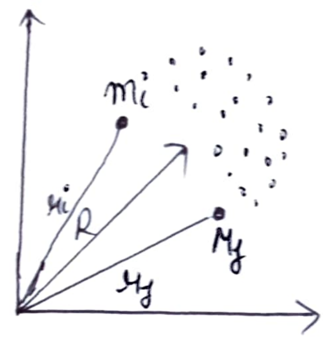
$$\vec{F} = \frac{d\vec{P}}{dt} = m\frac{d\vec{v}}{dt} = m\vec{a}$$

$$\boxed{\vec{F} = m\frac{d^2r}{dt^2}} \quad \text{——— (8)}$$

This is eqⁿ of motion of the particle

Mechanics of a system of Particle

Let us consider a system of N-Particles on which external force $\vec{F}$. Let ith particles of system having mass m_i and position vector r_j. Position of centre of mass depends on



i) Shape of system

ii) Distribution of mass in system

Position vector of centre of mass of the particle is given by.

$$\vec{R} = \frac{\sum_i^N m_i \vec{r_i}}{\sum_i^N m_i}$$

$$\left[\because \sum_i^N m_i = M \text{ (Total Mass)}\right]$$

$$R = \frac{1}{M}\sum_i m_i \vec{r_i}$$

$$\sum m_i \vec{r_i} = M\vec{R}$$

Total force acting on ith particle of system.

$$\vec{F_i} = \vec{F_i}^{(e)} + \sum_{\substack{j=1 \\ j\neq i}}^{N} \vec{F}_{ij} \quad \text{------} \quad (1)$$

By Newton's II^{nd} law

$$F_i = \frac{d\vec{P_i}}{dt} \quad \text{------} \quad (2)$$

Put the value of eqⁿ (2) in eqⁿ (1)

$$\frac{d\vec{P}}{dt} = \vec{F}_i^{(e)} + \sum_{\substack{J=1, \\ J \neq i}}^{N} \vec{F}_{ji} \quad \text{------} \quad (3)$$

Where, $\vec{F}_i^{(e)}$ is the external force acting on ith particle, and $\vec{F}_{ji}$ is the internal forces on the ith particle due to jth particle.

According to Newton's III^{rd}. law

$$\vec{F}_{ij} = -\vec{F}_{ji} \quad \text{------} \quad (4)$$

Motion of the whole system sum up. eqⁿ (4).

$$\sum_i \left(\sum_j \vec{F}_{ji} \right) + \sum \vec{F}_i^{(e)} = \sum \frac{d\vec{P_i}}{dt} \quad \text{------} \quad (5)$$

Now,
$$\sum_i \frac{d\vec{P_i}}{dt} = \sum_i \frac{d}{dt} (m_i \vec{v_i})$$

$$\sum_i \frac{d\vec{P_i}}{dt} = \sum_i m_i \frac{d\vec{v_i}}{dt} = \sum_i m_i \frac{d(m_i r_i)}{dt}$$

$$= \sum_i m_i \frac{d}{dt}\left(\frac{d\vec{r}_i}{dt}\right)$$

$$= \sum_i m_i \frac{d^2\vec{r}_i}{dt^2}$$

$$= \sum_i \frac{d^2(m_i \vec{r}_i)}{dt^2}$$ [$\because$ Assume m_i Constant]

$$= \frac{d^2}{dt^2} \sum_i (m_i \vec{r}_i)$$

Now, eq^n (5) can be written as,

$$\frac{d^2}{dt^2} \sum m_i \vec{r}_i = \sum_i \vec{F}_i^{(e)} + \sum_{\substack{i,j \\ i \neq j}} \vec{F}_{ji} \quad \text{-----} \quad (6)$$

Where, $\sum_{i,j}$ stands for $\sum_i \cdot \sum_j$

Now,

$$\vec{F}^{(e)} = \sum_i \vec{F}_i^{(e)} \quad \text{-----} \quad (7)$$

and

$$\sum_{\substack{i,j \\ i \neq j}} \vec{F}_{ji} = \sum_{\substack{i,j \\ i \neq j}} \vec{F}_{ij}$$

$$= \frac{1}{2} \sum_{\substack{i,j \\ i \neq j}} (\vec{F}_{ji} + \vec{F}_{ij}) = 0 \quad \text{-----} \quad (8)$$

Put the value of eq^n (7) and (8) in eq^n (6)

$$\boxed{\frac{d^2}{dt^2}\sum_i (m_i \vec{r}_i) = \vec{F}^{(e)}} \quad \text{--- --- (9)}$$

This eqⁿ represent the eqⁿ of motion of a system of particles.

Conservation Theorem for linear momentum

It states that "If total force $\vec{F}$ acting on the particle is zero, then linear momentum $(\vec{P})$ of the given particle is Conserved"

Proof →

Let force $\vec{F}$ acts on a particle then according to Newton's 2nd law of motion we have

$$\vec{F} = \frac{d\vec{P}}{dt} \quad \text{--- --- (2)}$$

Now if $\vec{F} = 0$ then from eqⁿ (2)

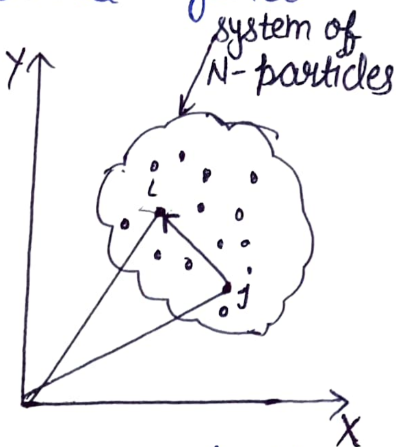
$$\frac{d\vec{P}}{dt} = 0$$

$$\boxed{P = \text{Constant}}$$

Hence Proved

Conservation theorem for linear momentum for system of particles

It states that "If sum of external force on the system particles is zero then total linear momentum of the system is constant or conserved



Proof – Let us consider a system of N-particles & take ith particle of the system. The total force acting on ith particle is given by

$$\vec{F}_1 = \vec{F}_i^{(e)} + \sum_{\substack{J=1 \\ i \neq J}}^{N} \vec{F}_{ij} \quad \text{—— (1)}$$

Now eq^n of motion of ith particle is given by according to Newton's law

$$\vec{F}_i = \frac{d\vec{P}_i}{dt} \quad \text{—— (2)}$$

Comparing eq^n (1) and (2)

$$\vec{F}_i^{(e)} + \sum_{\substack{J=1 \\ i \neq J}}^{N} \vec{F}_{ij} = \frac{d\vec{P}_i}{dt} \quad \text{—— (3)}$$

The eq^n of motion for whole system can be obtained by multiplying eq^n ③ both sides by $\sum_{i=1}^{N}$

$$\sum_{i=1}^{N} \vec{F}_i^{(e)} + \sum_{i=1}^{N} \sum_{j=1}^{N} \vec{F}_{ij} = \sum_{i=1}^{N} \frac{dP_i}{dt} \quad \text{——— ④}$$

$$F_i^{(e)} + \sum_{i,j} \vec{F}_{ij} = \frac{d}{dt} \sum_{i=1}^{N} \vec{P}_i \quad \text{——— ⑤}$$

$$\left[\because \sum_{i=1}^{N} \sum_{j=1}^{N} = \sum_{i,j} \right]$$

According to Newton's 3rd law for

$$\vec{F}_{ij} = -\vec{F}_{ji}$$

$$\boxed{\vec{F}_{ij} + \vec{F}_{ji} = 0}$$

$\sum_{i,j} \vec{F}_{ij} = 0$ · sum of internal for system of particles ——— ⑥

Using ⑥ in ⑤

$$\sum_{i=1}^{N} \vec{F}_i^{(e)} = \frac{d}{dt} \sum_{i=1}^{N} P_i \quad \text{——— ⑦}$$

$$\sum_{i=1}^{N} \vec{F}_i^{(e)} = \frac{d}{dt} \sum_{i=1}^{N} \vec{P}_i \quad \text{——— ⑧}$$

$$\sum_{i=1}^{N} \vec{F}_i^{(e)} = \frac{dP}{dt}$$

Where

$P = \sum_{i=1}^{N} \vec{P}_i$ = Total linear momentum for whole system.

Conservation Theorem for the Angular momentum of a particle

If total torque $\vec{\tau}$ acting on a particle is zero then the angular momentum (l) is conserved.

Proof - The angular momentum $\vec{l}$ of a particle P of mass m about any fixed point O, is defined as -

$$\vec{l} = \vec{r} \times \vec{p} \quad \text{———} \quad (1)$$

Where, $\vec{r}$ = radius vector
$\vec{P}$ = linear momentum

$$\vec{\tau} = \vec{r} \times \vec{F}$$
$$= \vec{r} \times \frac{d\vec{P}}{dt}$$

$$= \vec{r} \times \frac{d}{dt}(m\vec{v}) \qquad \text{——— (2)}$$

Taking total time derivatives of the quantities on R.H.S of eq^n (1)

$$\frac{d}{dt}(\vec{r} \times \vec{p}) = \vec{r} \times \frac{d\vec{p}}{dt} + \frac{d\vec{r}}{dt} \times \vec{p}$$

$$= \vec{r} \times \frac{d}{dt}(m\vec{v}) + \vec{v} \times m\vec{v}$$

$$\left(\because \frac{d\vec{r}}{dt} = \vec{v}\right)$$

Here

$$\vec{v} \times m\vec{v} = 0$$

and

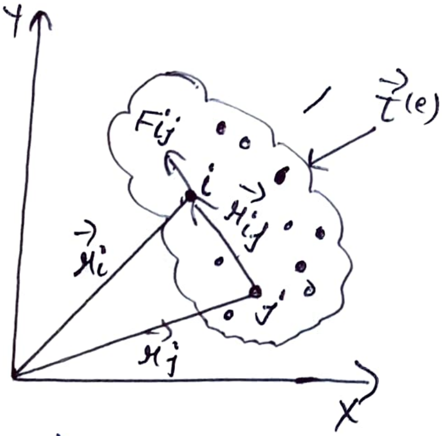
$$\frac{d}{dt}(\vec{r} \times \vec{p}) = r \times \frac{d}{dt}(mv)$$

$$\tau = \frac{d}{dt}(\vec{r} \times \vec{p}) \qquad [\text{using eq}^n \text{ (2)}]$$

$$\tau = \frac{d}{dt}(\vec{l}) \qquad \text{——— (3)}$$

Conservation theorem for total Angular momentum of a system of Particles

It states that "If total external torque acting on a system of particles is zero, then total angular momentum of the system is always constant.



Proof ⇒ Let us consider a system of particles. Let $\vec{l}_i$ be linear momentum of ith particle of the system having position vector $\vec{r}_i$. The angular momentum of ith particle about origin O is given by.

$$\vec{l}_i = \vec{r}_i \times \vec{p}_i \qquad \text{—— (1)}$$

Total angular momentum $\vec{\tau}$ of the system of particles about O is obtained by taking summation of eqn (1)

$$\sum_{i=1}^{N} l_i = \Sigma \vec{r}_i \times \vec{p}_i$$

$$\tau = \sum_{i=1}^{N} \vec{r_i} \times \vec{p_i} \quad \text{--- (2)}$$

Diff. both sides w.r.t. t

$$\frac{dL}{dt} = \frac{d}{dt} \sum_{i=1}^{N} (\vec{r_i} \times \vec{p_i})$$

$$= \sum_{i=1}^{N} \frac{d}{dt} (\vec{r_i} \times \vec{p_i})$$

$$= \sum_{i=1}^{N} \left(\vec{r_i} \times \frac{dP_i}{dt} + \frac{d\vec{r_i}}{dt} \times p_i \right)$$

$$= \sum_{i=1}^{N} \vec{r_i} \times \frac{d\vec{P_i}}{dt} + \vec{v_i} \times m\vec{v_i}$$

$$= \vec{r_i} \times \frac{dP_i}{dt} + m(\vec{v_i} \times \vec{v_i})$$

$$= \sum_{i=1}^{N} \vec{r_i} \times \frac{dP_i}{dt}$$

The eq^n motion of ith particle is given by

$$\vec{F_i} = \frac{d\vec{P_i}}{dt}$$

$\vec{F_i}$ = external + internal force

$$= \vec{F}_i^{(e)} + \sum_{\substack{j=1 \\ i \neq j}}^{N} \vec{F}_{ij}$$

$$\vec{F}_i^{(e)} + \sum_{j=1}^{N} \vec{F}_{ij} = \frac{d\vec{P_i}}{dt}$$

The eq^n of motion for the whole system Can obtained by taking summation eq^n 3

$$\sum_{i=1}^{N} F_i^{(e)} + \sum^{N} \sum^{N} \vec{F}_{ij} = \sum_{i=1}^{N} \frac{d\vec{P}_i}{dt}$$

$$\sum^{N} \vec{F}_i^{(e)} + \sum_{i,j}^{N} \vec{F}_{ij} = \sum_{i=1}^{N} \frac{d\vec{P}_i}{dt} \quad \text{——— } (4)$$

Taking cross Product of each terms of eq^n 4 by $\vec{r}_i$

$$\sum_{i=1}^{N} \vec{r}_i \times \vec{F}_i^{(e)} + \sum_{i,j}^{N} \vec{r}_i \times \vec{F}_{ij}$$

$$= \sum_{i=1}^{N} \vec{r}_i \times \frac{d\vec{P}_i}{dt} \quad \text{——— } (5)$$

Comparing 2 and 5

$$\sum_{i=1}^{N} \vec{r}_i \times \vec{F}_i^{(e)} + \sum_{i,j}^{N} \vec{r}_i \times \vec{F}_{ij} = \frac{d\vec{L}}{dt}$$

$$\sum^{N} \vec{\tau}^{(e)} + \sum_{i,j} \vec{r}_i \times \vec{F}_{ij} = \frac{d\vec{L}}{dt} \quad \text{——— } (6)$$

$$\vec{\tau}^{(e)} = \sum_{i=1}^{N} \tau_i^{(e)}$$

Now, $\sum_{i,j}^{N} \vec{r}_i \times \vec{F}_{ij} = \sum_{i,j}^{N} \vec{r}_{ij} \times \vec{F}_{ji}$ ——— (7)

Now $\sum_{i,j}^{N} \vec{r}_i \times \vec{F}_{ij} = \frac{1}{2} \sum_{i,j}^{N} 2(\vec{r}_i \times \vec{F}_{ij})$

$$= \frac{1}{2} \sum_{i,j}^{N} [\vec{r}_i \times \vec{F}_{ij} - \vec{r}_j \times \vec{F}_{ij}]$$

$$\sum_{i,j}^{N} \vec{r}_i \times \vec{F}_{ij} = \frac{1}{2} \sum_{i,j}^{N} (\vec{r}_i - \vec{r}_j) \times \vec{F}_{ij} \quad \text{—— (8)}$$

$$= \frac{1}{2} \sum_{i,j}^{N} \vec{r}_{ij} \times \vec{F}_{ij}$$

[$\because$ $\vec{r}_{ij}$ and $\vec{F}_{ij}$ are linear vector]

$$= \frac{1}{2} \sum_{i,j}^{N} (0)$$

$$\sum_{i,j}^{N} \vec{r}_i \times \vec{F}_{ij} = 0 \quad \text{—— (9)}$$

Using (9) in (6)

$$\tau^{(e)} + 0 = \frac{dL}{dt}$$

$$\tau^{(e)} = \frac{d\vec{L}}{dt}$$

If $\vec{\tau}^{(e)} = 0$

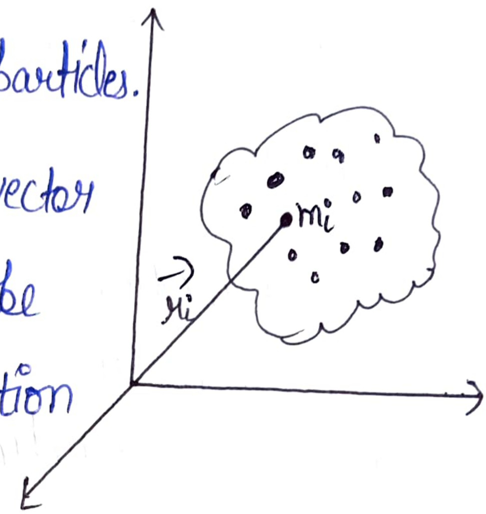
$$\frac{d\vec{L}}{dt} = 0$$

$$\boxed{\vec{L} = \text{constant}}$$

Centre of mass and equation of motion

The Centre of mass of a system of particles can be defined as a point where if the entire mass the system is supposed to the concentrated.

consider a system of N- particles. Let $\vec{r_i}$ be the position vector of ith particle and m_i be its mass, then eqn of motion for the particle is given by newton's second law of motion



$$\vec{F_i} = \frac{d\vec{p_i}}{dt} = \frac{d(m_i\vec{v_i})}{dt}$$

$$= \frac{d\,m_i}{dt}\frac{d\vec{r}}{dt} \qquad \left[\because \vec{V_i} = \frac{d\vec{r_i}}{dt}\right]$$

$$= \frac{d^2}{dt^2} m_i\vec{r_i} \quad \text{———} \quad (1)$$

For whole system, eq^n (1) can be written as.

$$\sum_{i=1}^{N} \vec{F_i} = \sum_{i=1}^{N} \frac{d^2}{dt^2} m_i\vec{r_i}$$

$$\vec{F} = \frac{d^2}{dt^2} \sum_{i=1}^{N} m_i r_i \quad (2)$$

Where, $\vec{F} = \Sigma \vec{F_i}$ = Total external force on the system.

$$\vec{F} = M \frac{d^2}{dt^2} \sum_{i=1}^{N} \frac{m_i \vec{r_i}}{M} \quad (3)$$

Where, $M = \sum_{i=1}^{N} m_i$ = total mass of the system

Constraints and Constrained motion -

Constraints -

The restriction or condition of imposed on the motion of a particle or system of particle are known as Constraints. Hence constraints limit the motion of a particle or system of particles.

Contrained Motion -

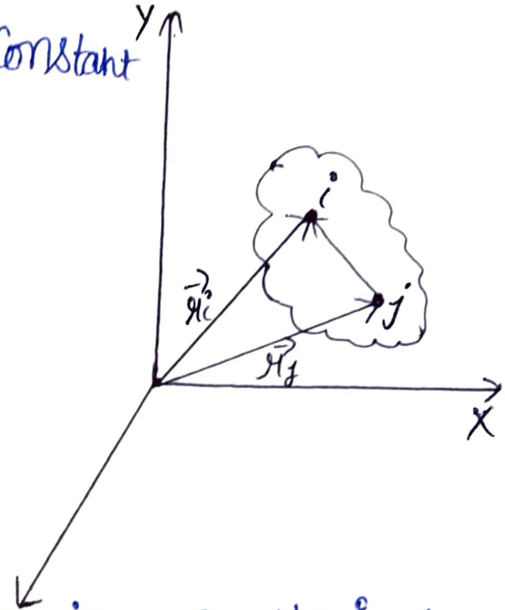
A motion which can't proceed in any arbitary manner and always Certain given Condition is known as Constrained motion.

Example ⟹

1) The motion of a rigid body is a constrained motion as distance b/w any 2 particles remains constant

$|\vec{r}_i - \vec{r}_j| = |k_{ij}|$

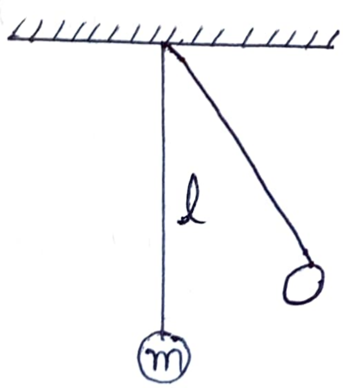
$$\boxed{|\vec{r}_i - \vec{r}_j| - |k_{ij}| = 0}$$



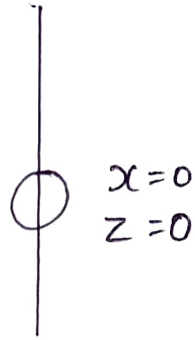
2) The motion of a point mass of angular momentum is constrained because is.

To following 2 Conditions -

a) The point mass always remains at a Constant distance from the point of suspension or length of the strong always and remains constants.

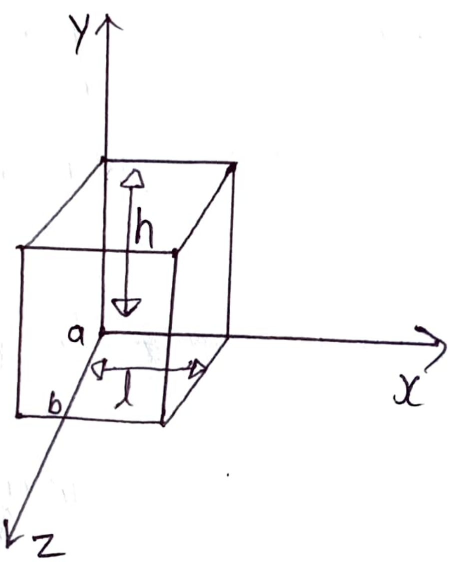


2) The motion of a bead sliding wire is simplest example of Constrained motion.



3) A particle in a box

$x \leq l$
$z \leq b$
$y \leq h$



Types of Constraints –

These are basically 4 types of contraints

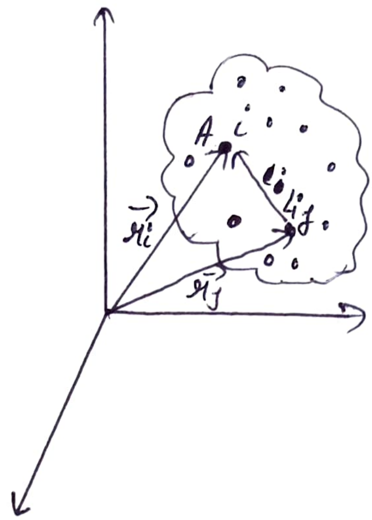
1) Holonomic contraints –

<u>Example</u> – The motion of the rigid body is a Contrained motion as the distance b/w any 2 pairs of its particles remain constant.

$$\vec{OB} + \vec{BA} = \vec{OA}$$

$$\vec{r_j} + \vec{C_{ij}} = \vec{r_i}$$

$$C_{ij} = \vec{r_i} - \vec{r_j}$$



2) Non-Holonomic constraints-

Non-holonomic constraints are constraints that cannot be integrated, and some examples of them include:

A unicyclist

The motion of a unicyclist is an example of a nonholonomic constraint because the unicyclist can point in any dirn for any position.

A car driving by plane

The velocity constraint of a car driving on a plane is a non-holonomic constraint because it cannot be integrated to give an equivalent configuration constraint.

3) Scleronomic and shenomic constraints -

The constraints which are independent of time Are called scleronomic constraint. The constraints which contain time explicity. They are called Phenomic constraints.

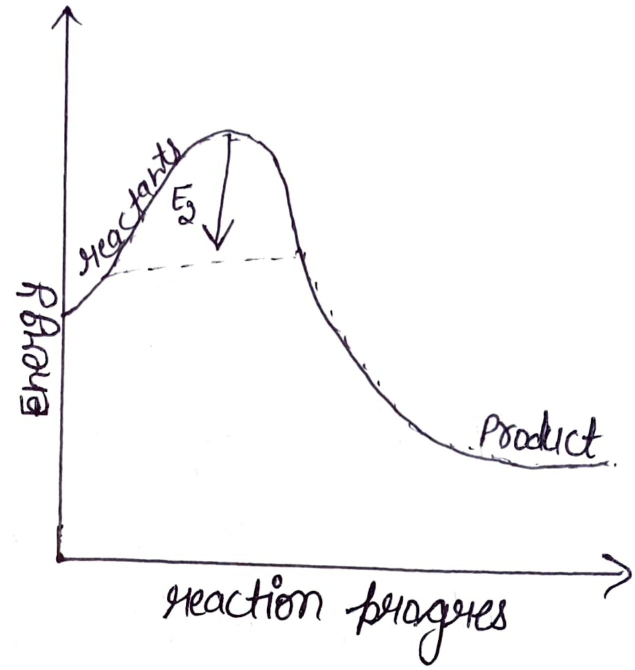
Example → A bead sliding on a rigid curved wire fixed in space. Is obviously subjected to scleronomic constraints And if the wire is moving in some prescribes fashion the constraints Become shenomous it is to be noted that if the wire moven say as a reaction to the bead is motion. Then the time dependence of the constraints enters in the

Potential Energy →

The energy that is stored in an object due to its position relative to some zero position.
An object possesses gravitational potential energy if it positioned at a height

above the zero height.

Energy diagram



Stable equilibrium and unstable equilibrium

Stable equilibrium —

$U(x) = $ P.E of system

$$\frac{dU(x)}{dx} = 0$$

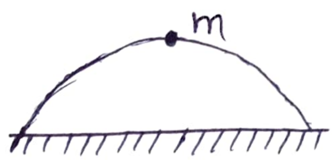
For U_{min} or U_{max}

For U_{min} $U''_{(x)} > 0$

Unstable equilibrium —

$$U(x), \frac{dU(x)}{dx} = 0$$

U_{max}, $U''(x) < 0$



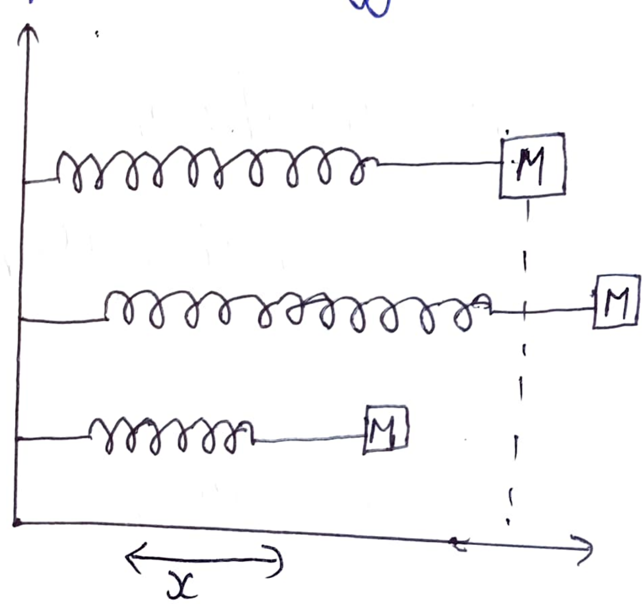
Elastic potential energy —

It is the energy stored in an elastic object when it is deforemed by an external force.

Or

Elastic potential energy is potential energy stored in an elastic material when it is strecched or compressed.

Expression for elastic potential energy



Form. the above diagram, it is clear that restoring force of the spring is directly proportional to displacement

That is,

$$F \propto S(x)$$
$$F = -kx$$

[where -ve sign represent the restoring force is opposite to dirn of displacement]

$$F = \frac{k}{|x|}$$

where, k is called spring constant.

The small amount of work done on the spring to small displacement dx is given by

$$dw = F.dx \quad \text{——} \quad (1)$$

∴ The total workdone on the string from initial to final position is:

$$x = 0 \longrightarrow x = x$$

$$\int dw = \int_0^x F.dx \quad \text{——} \quad (2)$$

We know that applied force is given by kx.

Now
eqn (2) is

$$\int dW = \int_0^x kx \cdot dx$$

$$W = k\int_0^x x \cdot dx$$

$$= k\left[\frac{x^2}{2}\right]_0^x$$

$$= \frac{k}{2}\left[x^2\right]_0^x$$

$$W = \frac{1}{2}kx^2 \quad \text{—— (3)}$$

This amount of work done is stored in the form of elastic energy

$$W = U$$

$$\boxed{U = \frac{1}{2}kx^2} \quad \text{—— (4)}$$

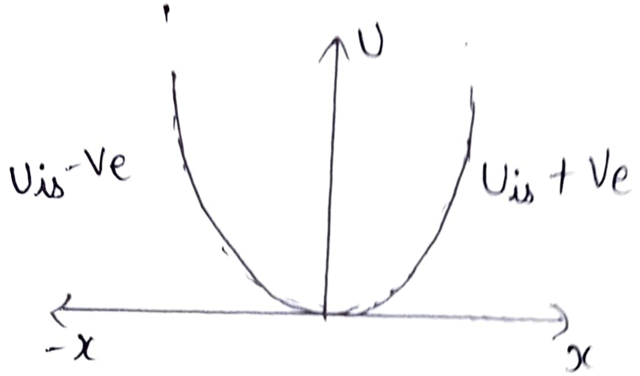
U = elastic potential energy
k = spring contant / force Constant
x = displacement

The variation of potential energy with displacement is shown in the following graph.

Restoring force

$$U = \frac{1}{2} k x^2$$

$$U \propto x^2$$



Force as Gradient of Potential energy

Let us consider a particle acted on by a Conservative force '$\vec{F}$' with corresponding potential energy 'u'.

$$\Delta u = -W$$

for differential path

$$W = \vec{F} \cdot d\vec{x}$$

$d\vec{x}$ = differential

$$dU = -F \cdot dr$$

$$\boxed{F = -\frac{du}{dr}}$$

$\frac{\partial U}{\partial x}$ → Partial diff. w.r.t. x putting y and ~~x~~ Constant

$\frac{\partial U}{\partial x}$ ⇒ Partial diff. w.r.t. ~~x~~ putting y and x Constant

$\frac{\partial U}{\partial y}$ ⇒ Partial diff w.r.t. y putting x, z constant

only for x direction

$$F_x = \frac{-\partial u}{\partial x} \quad \text{--- (1)}$$

only for y direction

$$F_y = \frac{-\partial u}{\partial y} \quad \text{--- (2)}$$

only for z direction

$$F_z = \frac{\partial x}{\partial z} \quad \text{--- (3)}$$

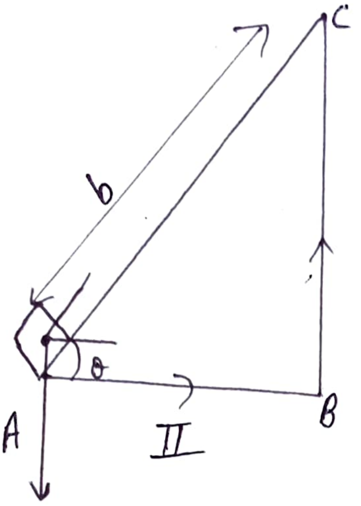
It motion is is 3- dimension

$$\vec{F} = F_x \hat{i} + F_y \hat{j} + F_z \hat{k} \quad \text{--- (4)}$$

Put in eq^n (4)

$$\vec{F} = -\left\{ \frac{\partial u}{\partial x} \hat{i} + \frac{\partial u}{\partial y} \hat{j} + \frac{\partial u}{\partial z} \hat{k} \right\}$$

Work done by Conservative force



$F = mg$

$W = Fs\cos\theta$

$(W_{AC})_{path-1} = mgs\cos(90+\theta)$

$(W_{AC})_{path-1} = -mgs\sin\theta$ —— ①

$(W_{AC})_{path-2} = W_{AB} + W_{BC}$

$= mg\,s\cos\theta\,\cos 90^\circ + mgs\sin\theta\cos 180$

$= -mgs\sin\theta$ —— ②

$= (W_{AC})_{path-2}$

$(W_{AC})_{path-1} = (W_{AC})_{path-2}$

$W_{AB} = 0,\ W_{BC} = -mgs\sin\theta$

$W_{ABCA} = W_{AB} + W_{BC} + W_{CA}$

$= 0 - mgs \sin\theta + mgs \cos(90-\theta)$

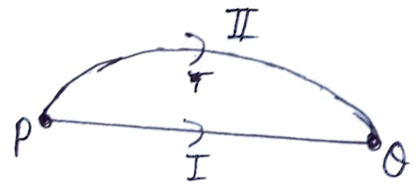
$= 0$

Conservative force :-

A force is said to be Conservative if the work done by it in moving a particle from one point to another point in space does not depend upon the actual path allowed by the particle and depends only on the initial and final positions of these points. If the particle moves from point P to point Q under the action of a Conservative force $\vec{F}$ then 3- alternative paths labelled as I, II & III, the work done from P to Q is same

$$W_{PQ} = \int_P^Q \vec{F}\cdot\vec{ds} = \int_P^Q \vec{F}\cdot\vec{ds}$$

along pat I



F

For a closed path the initial and final points coincide and hence work done by a conservative force around such as closed path will be zero.

$W = Fs\cos\theta = \vec{F}\cdot\vec{s}$

i.e.

$$\oint \vec{F}\cdot\vec{ds} = 0$$

Examples →

Gravitional force, electrosted Magnetic force, elastic force.

Unit - 2

Generalized Notations:

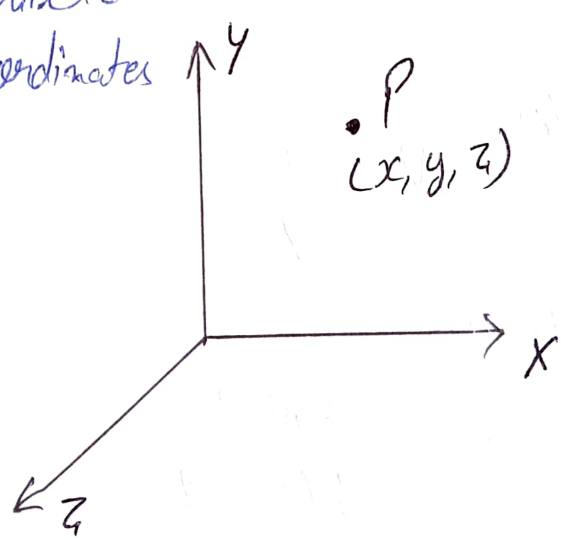
Degree of freedom -

The degree of freedom of a mechanical system is the minimum no. of independent co-ordinates required to completely describe its motion.

For example -

1) When a single particle moves in space, it has 3 degree of freedom.

Here, P is any particle which having co-ordinates (x, y, z).

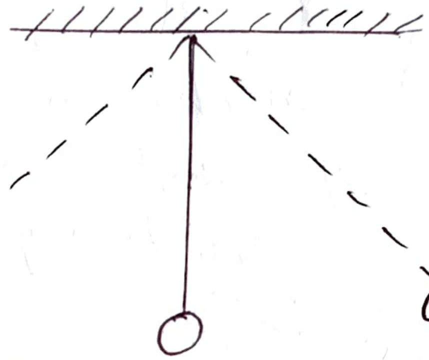


2) A bird flying in sky.

* A system of N - particles free from constraints has 3N - degree of freedom.

* A system of N - particles subjected to K constraints then $f = 3N - K$ degree of freedom.

3) Motion of simple pendulum.

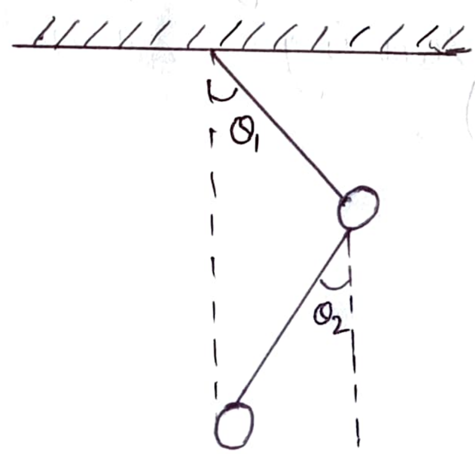


Here, $N = 1$, $K = 2$

$f = 3N - K = 3(1) - 2 = 3 - 2 = 1$

$\boxed{f = 1}$

4) Motion of double Pendulum.



Here,

$N = 2$

$K = 4$

then

$f = 3N - K$

$= 3(2) - 4$

$= 6 - 4$

$= 2$

$\boxed{f = 2}$

$$\boxed{f = 3N - K}$$

These f no. of minimum independent co-ordinates required to describe configuration and motion of a mechanical system are called Generalized co-ordinates. and are denoted by $q_1, q_2, \ldots q_f$ or just by

$$q_i \; (i = 1, 2, \ldots, f)$$

We can express cartesian co-ordinates $\vec{r}_i$ in term of generalized co-ordinates and time in the form of eq.^{n}

$$r_i = \vec{r}_i(q_1, q_2, \ldots, q_f, t)$$

Ex- If a particle of mass (m) is constrained to move on a circular wire in a plane, then motion can be describe by, Generalized co-ordinate(θ)

$$x = r\cos\theta$$
$$y = r\sin\theta$$

where, r is the radius of circle. Since r is constant the motion of particle can be specified by use of generalzied co-ordinate θ alone.

Generalized Displacement -

Let us consider a N-particle system for which a small displacement $\delta\vec{r}_i$ is defined by change in position co-ordinates $\vec{r}_i, (i = 1, 2, 3 \ldots N)$ with time (t) kept as constant. The position vector $\vec{r}_i$ of the ith particle in the form of generalized co-ordinates can be written as.

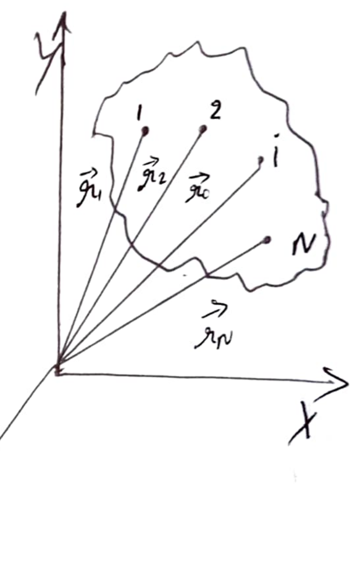
$$\vec{r}_i = \vec{r}_i(q_1, q_2, q_3 \ldots, q_f, t)$$

Using Euler's theorem

$$\delta\vec{r}_i = \frac{d\vec{r}_i}{dq_1}\delta q_1 + \frac{d\vec{r}_i}{dq_2}\delta q_2 + \ldots \frac{d\vec{r}_i}{dq_f}\delta q_f$$

$$\boxed{\delta\vec{r}_i = \sum_{k=1}^{3N} \frac{\partial r_i}{\partial q_k}\delta q_k}$$

where δq_k represent Generalized displacement



Generalized velocity

Let us consider a N-particle system in which the generalized velocity $\dot{q}_k$ is the time derivative of generalized co-ordinate q_k. The position vector $\vec{r}_i$ of the ith particle in the form of generalized co-ordinates and time (t) can be written as generalized co-ordinates and time (t) can be written as.

$$\vec{r}_i = \vec{r}_i(q_1, q_2 \dots q_f, t)$$

Using Euler's theorem, we have

$$\frac{d\vec{r}_i}{dt} = \frac{dr_i}{dq_1}\frac{dq_1}{dt} + \frac{dr_i}{dq_2}\frac{dq_2}{dt} + \dots \frac{dr_i}{dq_f}\frac{dq_f}{dt}) + \frac{dr_i}{dt}\frac{dt}{dt}$$

$$\vec{V}_i = \sum_{k=1}^{f} \frac{d\vec{r}_i}{dq_k}\frac{\partial q_k}{dt} + \frac{d\vec{r}_i}{dt}$$

$$\boxed{\vec{v}_i = \sum_{k=1}^{f} \frac{d\vec{r}_i}{dq_k}\dot{q}_k + \frac{d\vec{r}_i}{dt}}$$

where $\dot{q}_k$ is generalized velocity.

Generalized Acceleration -

The generalized acceleration is equal to the time derivative of the generalized velocity. Let us consider a system of N-particles. The velocity of ith particle of the system is given by.

$$\vec{V}_i = \dot{\vec{r}}_i = \sum_{k=1}^{f} \frac{dr_i}{dt}\dot{q}_k + \frac{d\vec{r}_i}{dt} \quad \text{—— (1)}$$

Diff. both sides w.r.t. time

$$\dot{\vec{V}}_i = \frac{d}{dt}\left[\sum_{k=1}^{f} \frac{\partial\vec{r}_i}{dt}\dot{q}_k + \frac{d\vec{r}_i}{dt}\right]$$

$$\vec{a}_i = \sum_{k=1}^{f} \frac{d\vec{r}_i}{dq_k}\dot{q}_k + \sum_{k=1}^{f} \frac{\partial\vec{r}_i}{dq_k}\ddot{q}_k + \frac{d\vec{r}_i}{dt} \quad \text{—— (2)}$$

Using eqⁿ ① and ②

$$\vec{a_i} = \sum_{K=1}^{f} \frac{d}{dq_K}\left[\sum_{J=1}^{f} \frac{d\vec{r_i}}{dq_j}\dot{q}_j + \frac{d\vec{r_i}}{dt}\right]\dot{q}_K + \sum_{K=1}^{f} \frac{d\vec{r_i}}{dq_K}\ddot{q}_K + \frac{d}{dt}\left[\sum_{J=1}^{f} \frac{d\vec{r_i}}{dq_j}\dot{q}_J + \frac{d\vec{r_i}}{dt}\right]$$

$$\vec{a_i} = \sum_{K=1}^{f}\sum_{J=1}^{f} \frac{d^2 r_i}{dq_K\, dq_J}\dot{q}_J\dot{q}_K + \sum_{K=1}^{f} \frac{d^2 r_i}{\partial q_K\, dt}\dot{q}_K + \sum_{J=1}^{f} \frac{d^2 r_i}{dt\, dq_J} + \frac{d^2 r_i}{dt^2}$$

which is required expression for generalised acceleration.

Generalised Momentum -

Kinetic energy of the ith particle moving velocity v_i is given by

$$T = \frac{1}{2} m_i v_i^2$$

$$T = \frac{1}{2} m_i \left(\frac{dx_i}{dt}\right)^2 = m_i \dot{x}_i^2 \quad \text{—— ①}$$

$$\left[\because v_i = \frac{dx_i}{dt} = \dot{x}_i\right]$$

Partially diff. eqⁿ ① w.r.t. $\dot{x}_i$, we get

$$\frac{dT}{d\dot{x}_i} = \frac{1}{2} m_i \cdot 2\dot{x}_i$$

$$\frac{dT}{d\dot{x}_i} = m_i \dot{x}_i \quad \text{—— ②}$$

$$T = \sum_{i=1}^{N}\sum_{j=1}^{3N}\sum_{k=1}^{3N} \frac{1}{2} m_i \frac{d\vec{r_i}}{dq_j} \frac{dr_i}{dq_k} q_j \dot{q}_k +$$

$$2\sum_{i=1}^{N}\sum_{k=1}^{3N} \frac{1}{2} m_i \frac{d\vec{r_i}}{dq_k} \frac{dr_i}{dt} \dot{q}_k + \frac{1}{2}\sum_{i=1}^{N} \frac{1}{2} m_i \left(\frac{dr_i}{dt}\right)^2$$

Partially diff. of above eqn

$$\frac{dT}{d\dot{q}_k} = \sum_{i=1}^{N}\sum_{j=1}^{3N} \frac{1}{2} m_i \frac{d\vec{r_i}}{dq_j} \frac{d\vec{r_i}}{dq_k} q_j + \sum_{j=1}^{N} m_i \frac{d\vec{r_i}}{dq_k} \frac{d\vec{r}}{dt}$$

We have $P_K = \frac{dT}{d\dot{q}_k}$

$$P_K = \sum_{i=1}^{N}\sum_{j=1}^{3N} \frac{1}{2} m_i \frac{d\vec{r_i}}{dq_j} \frac{d\vec{r_i}}{dq_K} q_j + \sum_{i=1}^{N} M_i \frac{d\vec{r_i}}{dq_K} \frac{dr_i}{dt}$$

Generalized Potential :-

If the force acting on the system are derived from a scalar potential V depending on the position only, then V represents the potential energy of the system.

The work done by the force acting in an displacement $\delta \vec{r_i}$ of the system is

$$\delta W = -\delta V$$
$$= -\sum\left(\frac{\partial V}{\partial x_i}\delta x_i + \frac{\partial V}{\partial y_i}\delta y_i + \frac{\partial V}{\partial z_i}\delta z_i\right)$$

$$\delta W = \sum_{k=1}^{3N} Q_k \delta q_k$$

Thus, $Q_k = -\frac{\partial V}{\partial q_k}$

$$\frac{\partial V}{\partial q_k} = \sum_{i=1}^{N}\left(\frac{\partial V}{\partial x_i}\frac{\partial x_i}{\partial q_k} + \frac{\partial V}{\partial y_i}\frac{\partial y_i}{\partial q_k} + \frac{\partial V}{\partial z_i}\frac{\partial z_i}{\partial q_k}\right)$$
$$= -\sum_{i=1}^{N}\left[\vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_k}\right]$$

$$= -Q_k \qquad (2)$$

If the system is not conservative, i.e. it also depends on the generalized velocities $\dot{q}$ besides q, we then define generalized force Q_j associated with a co-ordinate q_j not by eqn (2) but by

$$Q_j = -\frac{\partial U}{\partial q_j} + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_j}\right) \qquad (3)$$

where $U(q_i, \dot{q}_i)$ is called a velocity dependent potential or 'generalized potential

Generalized force:-

If the force acting on the system are derived from a scalar potential V depending on the position only, then V represents the potential energy of the system.

The work done by the force on the system in an arbitrary displacement $\delta \vec{r}_i$ of the system is.

$$\delta W = -\delta V = -\sum_{i=1}^{N}\left(\frac{\partial V}{\partial x_i}\delta x_i + \frac{\partial V}{\partial y_i}\delta y_i + \frac{\partial V}{\partial z_i}\delta z_i\right) \quad \text{---} (1)$$

Also $\delta W = -\delta V = -\sum_{K=1}^{3N}\frac{\delta V}{\delta q_K}\delta q_K = \sum_{K=1}^{3N} Q_K \delta q_K$

Thus $Q_K = -\frac{\partial V}{\partial q_K}$ ---------- (2)

Now from eq^n (14)

$$\frac{\partial V}{\partial q_K} = \sum_{i=1}^{N}\left(\frac{\partial V}{\partial x_i}\frac{\partial x_i}{\partial q_K} + \frac{\partial V}{\partial y_i}\frac{\partial y_i}{\partial q_K} + \frac{\partial V}{\partial z_i}\frac{\partial z_i}{\partial q_K}\right)$$

$$= -\sum_{i=1}^{N}\left[\vec{F}_i \frac{\partial \vec{r}_i}{\partial q_K}\right] = -Q_K$$

[from 2nd eq^n]

We define the generalized force Q_j associated with a co-ordinate q_j not by eq^n (2) but by

$$Q_j = -\frac{\partial U}{\partial q_j} + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_j}\right) \quad \text{———} (3)$$

$$\delta W = \sum_{i}^{N} \vec{F}_i \cdot \delta \vec{r}_i$$

$$= \sum_{i=1}^{N} \vec{F}_i \cdot \sum_{j=1}^{3N} \frac{\partial \vec{r}_i}{\partial q_j} \delta q_j$$

$$= \sum_{i=1}^{N} \left[\sum_{j=1}^{3N} \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} \right] \delta q_j$$

$$= \sum_{i=1}^{3N} Q_j \, \delta q_j$$

where,
$$Q_j = \sum_{i=1}^{N} \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}$$

Q_j is called the generalized force associated with a co-ordinate q_j.

If we consider again particle defined by (r, θ) generalised co-ordinates, the components of force are given by -

$$\vec{F} = F_r \hat{r} + F_\theta \hat{\theta}$$, $\hat{r}, \hat{\theta}$ are unit vectors in this case

The generalized force associated with r-co-ordinate and θ-co-ordinate are respectively

$$Q_r = \vec{F} \cdot \frac{\partial \vec{r}}{\partial r} = (F_r \hat{r} + F_\theta \hat{\theta}) \cdot \frac{\partial r}{\partial r} \hat{r} = F_r$$

$$Q_\theta = \vec{F} \cdot \frac{\partial \vec{r}}{\partial \theta} = (F_r \hat{r} + F_\theta \hat{\theta}) \cdot \left(r \frac{\partial \hat{r}}{\partial \theta} \right) = r F_\theta$$

Q_r is the component of force in r-dir^n and Q_θ is the torque acting on the particle to increase θ.

Transformation Equations

If x_i, y_i, z_i be the cartesian co-ordinates of ith particle of the system, then these cartesian co-ordinates can be expressed as functions of generalized co-ordinates

$$q_1, q_2, \ldots\ldots, q_f \text{ i.e.}$$

$$\left.\begin{aligned} x_i &= x_i(q_1, q_2, \ldots\ldots q_f, t) \\ y_i &= y_i(q_1, q_2, \ldots\ldots q_f, t) \\ z_i &= z_i(q_1, q_2, \ldots\ldots, q_f, t) \end{aligned}\right] \quad \text{—— (1)}$$

where t denotes the time.

If r_i be the position vector of the ith particle, i.e.;

$$r_i = \hat{i}x_i + \hat{j}y_i + \hat{k}z_i$$

where $\hat{i}$, $\hat{j}$ and $\hat{k}$ are the unit vectors along x, y, and z axis respectively then.

$$\vec{r}_i = \vec{r}_i(q_1, q_2, \ldots\ldots q_f, t) \quad \text{—— (2)}$$

which is the vector form of eq^n (1) where $i = 1$ to N. eq^n (1) and (2) are called transformation equations from the set of $\vec{r}_i \{\equiv (x_i, y_i, z_i)$ variables to the q_i set.

where $U(q_i, \dot{q}_i)$ is called a velocity dependent potential or simply 'generalized potential.

Velocity and acceleration in cylindrical co-ordinates.

Let P be a point in space such that its cartesian co-ordinates are (x, y, z). Draw a radius vector r from O'.

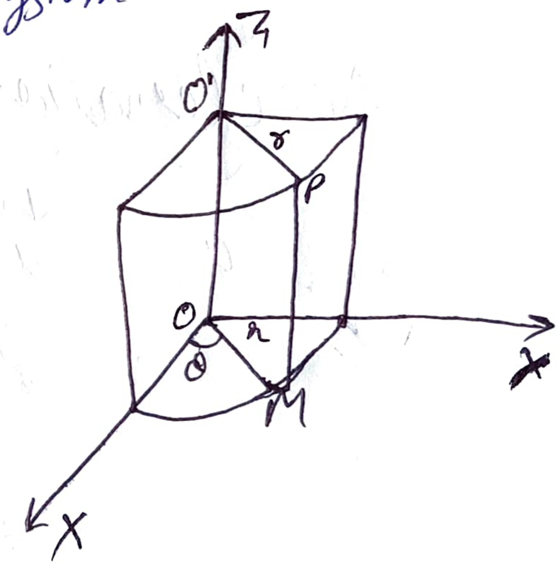
$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

(cartesian co-ordinate system)

$$V = \frac{d\vec{r}}{dt}$$

Cylindrical co-ordinate system

ρ, ϕ, z



* $x = \rho\cos\phi$

$y = \rho\sin\phi$

$z = z$

$$\hat{\rho} = \cos\phi\,\hat{i} + \sin\phi\,\hat{j}$$

we can find also $\hat{\rho} = \dfrac{\frac{\partial\vec{r}}{\partial\rho}}{\left|\frac{\partial\vec{r}}{\partial\rho}\right|}$

(13)

$\hat{\phi} = -\sin\phi\,\hat{i} + \cos\phi\,\hat{j}$; $\hat{\phi} = \dfrac{\frac{\partial \vec{r}}{\partial \phi}}{\left|\frac{\partial \vec{r}}{\partial \phi}\right|}$

$\hat{z} = \hat{k}$

$\vec{r} = \rho\cos\phi\,\hat{i} + \rho\sin\phi\,\hat{j} + z\hat{k}$ ---(2)

from eq^n (1) and (2)

$\vec{r} = \rho(\cos\phi\,\hat{i} + \sin\phi\,\hat{j}) + z\hat{k}$

$\vec{r} = \rho\hat{\rho} + z\hat{k}$ (cylindrical co-ordinate system)

* $\vec{v} = \dfrac{d\vec{r}}{dt} = \dfrac{d}{dt}(\rho\hat{\rho} + z\hat{k})$

$= \dot{\rho}\hat{\rho} + \rho\dfrac{d\hat{\rho}}{dt} + \dot{z}\hat{k}$

$= \dot{\rho}\hat{\rho} + \rho(\dot{\phi}\hat{\phi}) + \dot{z}\hat{k}$ $\left[\because \dfrac{d\hat{\rho}}{dt} = \dot{\phi}\hat{\phi}\right]$

$\therefore$

Acceleration

$\vec{a} = \dfrac{d\vec{v}}{dt} = \dfrac{d}{dt}(\dot{\rho}\hat{\rho} + \rho\dot{\phi}\hat{\phi} + \dot{z}\hat{k})$

$= \ddot{\rho}\hat{\rho} + \dot{\rho}\dfrac{d\hat{\rho}}{dt} + \dot{\rho}\dot{\phi}\hat{\phi} + \rho\ddot{\phi}\hat{\phi} + \rho\dot{\phi}\dfrac{d\hat{\phi}}{dt} + \ddot{z}\hat{k}$

$= \ddot{\rho}\hat{\rho} + \dot{\rho}\dot{\phi}\hat{\phi} + \dot{\rho}\dot{\phi}\hat{\phi} + \rho\ddot{\phi}\hat{\phi} + \rho\dot{\phi}(-\dot{\phi}\hat{\rho}) + \ddot{z}\hat{k}$

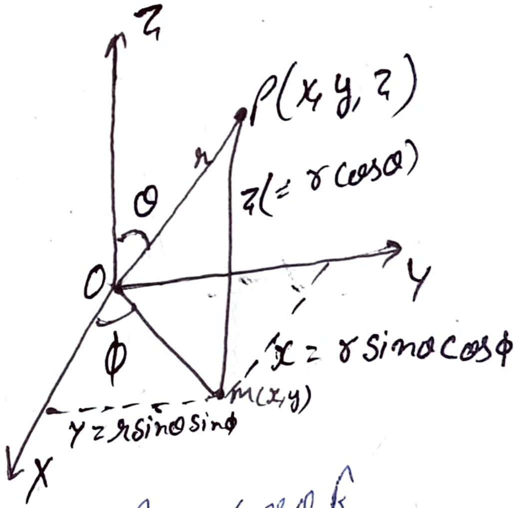
$$\vec{a} = \ddot{\rho}\hat{\rho} - \rho\dot{\phi}^2\hat{\rho} + 2\dot{\rho}\dot{\phi}\hat{\phi} + \rho\ddot{\phi}\hat{\phi}$$

Velocity in Spherical co-ordinates (r, θ, ϕ)

Let P be a point in space such that its cartesian co-ordinates are (x, y, z). Then spherical polar co-ordinates of point P.

$$\vec{OP} = \vec{r} = r\sin\theta\cos\phi\,\hat{i} + r\sin\theta\sin\phi\,\hat{j}$$

$$\vec{r} = r[\sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{i} + \cos\theta\,\hat{k}]$$



$$r = r\hat{e}_r$$

$$\vec{V} = \frac{d\vec{r}}{dt}$$

$$= \frac{d}{dt}[r\hat{e}_r]$$

$$= \dot{r}\hat{e}_r + r\dot{\hat{e}}_r$$

$$\hat{e}_r = \sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k}$$

$$\frac{d\hat{e}_r}{dt} = \cos\theta(\dot{\theta})\cos\phi\,\hat{i} + \sin\theta(-\sin\phi)\dot{\phi}\,\hat{i}$$

$$+ \cos\theta\dot{\theta}\sin\phi\,\hat{j} + \sin\theta(\cos\phi)\dot{\phi}\,\hat{j}$$

$$+ (1-\sin\theta)\dot{\theta}\,\hat{k}$$

$$\dot{\hat{e}}_r = \dot{\theta}\hat{e}_\theta + \sin\theta\dot{\phi}\hat{e}_\phi$$

$$\vec{V} = \dot{r}\hat{e}_r + r\dot{\hat{e}}_r$$

$V_r = \dot{r}$

$V_\theta = r\dot{\theta}$

$V_\phi = r\sin\theta\,\dot{\phi}$

$$V = \sqrt{V_r^2 + V_\theta^2 + V_\phi^2}$$

$$= \sqrt{(\dot{r})^2 + (r\dot{\theta})^2 + (r\sin\theta\,\dot{\phi})^2}$$

$$V = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta + r\sin\theta\,\dot{\phi}\,\hat{e}_\phi$$

Acceleration

$$a = \frac{dV}{dt} = \frac{d}{dt}[\dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta + r\sin\theta\,\dot{\phi}\,\hat{e}_\phi]$$

$$= \frac{d}{dt}[\dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta]$$

$$= \ddot{r}\hat{e}_r + \dot{r}\dot{\hat{e}}_r + \dot{r}\dot{\theta}\hat{e}_\theta + r\ddot{\theta}\hat{e}_\theta + r\dot{\theta}\dot{\hat{e}}_\theta \quad (2)$$

$$\Rightarrow \dot{\hat{e}}_r = \dot{\theta}\hat{e}_\theta \qquad \dot{\hat{e}}_\theta = -\dot{\theta}\hat{e}_r$$

Now, put the values $\dot{\hat{e}}_r$ & $\dot{\hat{e}}_\theta$ in eq^n (2)

$$\vec{a} = \ddot{r}\hat{e}_r + \dot{r}(\dot{\theta}\hat{e}) + \dot{r}\dot{\theta}\hat{e}_\theta + r\ddot{\theta}\hat{e}_\theta + r\dot{\theta}(-\dot{\theta}\hat{e}_r)$$

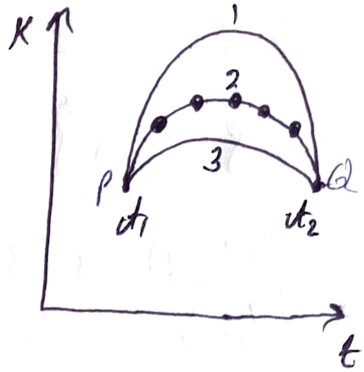
$$= (\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{e}_\theta$$

a_r (Radial component of Acceleration)

a_θ (Transverse component of acceleration)

Hamilton's Variational Principle -

It state that "actual path followed by a conservative holonomic, dynamical system over a fixed time interval (say b/w t_1 & t_2) is that over which the line integral of Lagrangian L is extremum (either max or min).



Mathematically,

$$I = \int_{t_1}^{t_2} L\,dt = \text{extremum} \quad \text{(either max or min.)} \longrightarrow (1)$$

Here I is known Action Integral & L is Lagrangian function which $L = T - V$

So from eq^n (1), variation in line integral i.e δI must be zero

$$\delta I = \delta \int_{t_1}^{t_2} L\,dt = 0 \longrightarrow (2)$$

Deduction -

We know that Hamilton's variational principal is

$$\delta I = \delta \int_{t_1}^{t_2} L\,dt = 0 \longrightarrow (1)$$

But we know that Hamiltion function 'H' is given by

$$H = \sum_i p_i \dot{q}_i - L \Rightarrow L = \sum p_i \dot{q}_i - H \quad — (2)$$

Using eq^n (2) in (1) we get

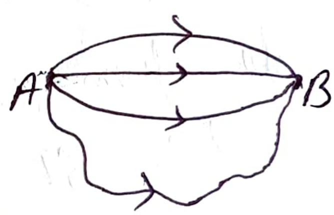
$$\delta I = \delta \int_{t_1}^{t_2} \sum_i (p_i \dot{q}_i - H)\, dt = 0 \quad — (3)$$

eq^n (3) is known as Hemiltion's principle which leads to

$$\delta I = \delta \int_{t_1}^{t_2} \left(\sum_i p_i \dot{q}_i - H \right) dt = 0 \quad — (3)$$

Let each of possible paths in the configurational space is labelled by a parameter α, then δ variation can be expressed as

$$\delta = d\alpha \cdot \frac{d}{d\alpha} \quad — (4)$$



Using (4) in (3) we get

$$d\alpha \frac{d}{d\alpha} \int_{t_1}^{t_2} \left(\sum_i p_i \dot{q}_i - H \right) dt = 0$$

$$d\alpha \frac{d}{d\alpha} \int_{t_1}^{t_2} \left(\sum_i p_i \dot{q}_i - H \right) dt = 0$$

$$d\alpha \int_{t_1}^{t_2} \left[\sum_i \frac{d}{d\alpha}(p_i \dot{q}_i) - \frac{d}{d\alpha} H(q_i, p_i, t) \right] dt = 0$$

$$d\alpha \int_{t_1}^{t_2} \left[\sum_i \frac{d}{d\alpha}(p_i \dot{q}_i) - \frac{d}{d\alpha} H(q_i; p_i, t) \right] dt$$

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$$d\alpha \int_{t_1}^{t_2} \left[\sum_i \frac{d}{d\alpha}(p_i \dot{q}_i) - \frac{d}{d\alpha} H(q_i, p_i, t) \right] dt$$

$$d\alpha \cdot \int_{t_1}^{t_2} \left[\sum_i \left(p_i \frac{d\dot{q}_i}{d\alpha} + \dot{q}_i \frac{dp_i}{d\alpha} - \frac{dH}{dq_i}\frac{dq_i}{d\alpha} - \frac{dH}{dp_i}\frac{dp_i}{d\alpha} \right) - \frac{dH}{dt} \cdot \frac{dt}{d\alpha} \right] dt = 0 \quad \text{——— (5)}$$

Since time travel along any path is same $\frac{dt}{d\alpha} = 0$ ——— (6)

Using (6) in (5) we get

$$d\alpha \int_{t_1}^{t_2} \left[\sum_i \left(p_i \frac{d\dot{q}_i}{d\alpha} + \dot{q}_i \frac{dp_i}{d\alpha} - \frac{dH}{dq_i} \cdot \frac{dq_i}{d\alpha} - \frac{dH}{dp_i} \cdot \frac{dp_i}{d\alpha} \right) \right] \cdot dt = 0 \quad \text{——— (7)}$$

Consider the Ist term of integral in eq^n (7)

$$\int_{t_1}^{t_2} p_i \frac{d\dot{q}_i}{d\alpha} dt = \int_{t_1}^{t_2} p_i \frac{d}{d\alpha}\left(\frac{d}{dt} q_i\right) dt$$

$$\int_{t_1}^{t_2} p_i \frac{d\dot{q}_i}{d\alpha} dt = \int_{t_1}^{t_2} p_i \frac{d}{dt}\left(\frac{dq_i}{d\alpha}\right) dt$$

$$= \left[p_i \int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{dq_i}{d\alpha}\right) dt - \int_{t_1}^{t_2} \left\{ \dot{p}_i \int \frac{d}{dt}\left(\frac{dq_i}{d\alpha}\right) dt \right\} dt \right.$$

$$\int_{t_1}^{t_2} p_i \frac{d\dot{q}_i}{d\alpha} dt = -\int_{t_1}^{t_2} \dot{p}_i \frac{dq_i}{d\alpha} dt \quad \text{——— (8)}$$

Using (8) in (7), we get

$$d\alpha \int_{t_1}^{t_2} \sum \left(-\dot{p}_i \frac{dq_i}{d\alpha} + \dot{q}_i \frac{dp_i}{d\alpha} - \frac{dH}{dq_i}\frac{dq_i}{d\alpha} - \frac{dH}{dp_i}\frac{dp_i}{d\alpha}\right) dt = 0$$

$$d\alpha \int_{t_1}^{t_2} \sum_i \left[\left(-\dot{p}_i - \frac{dH}{dq_i}\right)\frac{dq_i}{d\alpha} + \left(\dot{q}_i - \frac{dH}{dp_i}\right)\frac{dp_i}{d\alpha}\right] dt = 0$$

$$\boxed{\delta = d\alpha \frac{d}{d\alpha}}$$

$$\int_{t_1}^{t_2} \sum_i \left[\left(-\dot{p}_i - \frac{dH}{dq_i}\right) d\alpha \frac{dq_i}{d\alpha} + \left(\dot{q}_i - \frac{dH}{dp_i}\right) d\alpha \frac{dp_i}{d\alpha}\right] dt = 0$$

$$\int_{t_1}^{t_2} \sum_i \left[\left(-\dot{p}_i - \frac{\partial H}{\partial q_i}\right)\delta q_i + \left(\dot{q}_i - \frac{dH}{dp_i}\right)\delta p_i\right] dt = 0$$

As p_i & q_i are independent variables so their variations δp_i and δq_i will also be independent. Thus above integral can vanish only if their coefficients separately vanish i.e.

$$\boxed{\dot{p}_i = -\frac{dH}{dq_i}} \qquad \boxed{\dot{q}_i = \frac{dH}{dp_i}}$$

which are required Hamilton's canonical eq^n.

Lagrange's eq^n of motion Hamilton's variational principle

The Lagrangian function is given by

$$L = L(q_1, q_2, q_3 \ldots q_i ; \dot{q}_1, \dot{q}_2, \ldots \dot{q}_i, t)$$

If Lagrangian does not depend upon time explicity, then we have

$$\delta L = \Sigma \frac{\partial L}{\delta q_i} \delta q_i + \Sigma \frac{\partial L}{\delta \dot{q}_i} \delta \dot{q}_i$$

Derivation

According to Hamilton's variational principle

$$\delta I = \delta \int_{t_1}^{t_2} L\,dt = 0$$

$$\delta I = \delta \int_{t_1}^{t_2} [T(q_k, \dot{q}_k) - V(q_k)]\,dt = 0$$

$$= \int_{t_1}^{t_2} [\delta T(q_k, \dot{q}_k) - \delta V(q_k)]\,dt = 0$$

$$= \int_{t_1}^{t_2} \left[\frac{\partial T}{\partial q_k} \delta q_k - \frac{\partial V}{\partial q_k} \delta q_k) - \frac{\partial T}{\partial \dot{q}_k} \delta \dot{q}_k \right] dt = 0 \quad \text{--- (1)}$$

Integrating both sides with respect to t from t_1 to t_2, we get

$$\int_{t_1}^{t_2} \delta L\,dt = \int_{t_1}^{t_2} \Sigma \frac{\partial L}{\partial q_k} \delta q_k\,dt + \int_{t_1}^{t_2} \sum_k \frac{\partial L}{\partial \dot{q}_k} \delta \dot{q}_k\,dt$$

But acc. to Hamilton's principle

$$\delta \int_{t_1}^{t_2} L\,dt = 0$$

Therefore eq^n (2) becomes

$$\int_{t_1}^{t_2} \sum_k \frac{\partial L}{\partial q_k} \delta q_k \, dt + \int_{t_1}^{t_2} \sum_k \frac{\partial L}{\partial \dot{q}_k} \delta \dot{q}_k \, dt = 0 \quad \text{--- (3)}$$

Since $\delta \dot{q}_k = \delta\left(\frac{dq_k}{dt}\right) = \frac{d}{dt}(\delta q_k)$ then eq^n (3)

$$\int_{t_1}^{t_2} \sum \frac{\partial L}{\partial q_k} \delta q_k \, dt + \int_{t_1}^{t_2} \sum \frac{\partial L}{\partial \dot{q}_k} \frac{d}{dt}(\delta q_k) dt = 0$$

Integrating second term by parts,

$$\int_{t_1}^{t_2} \sum \frac{\partial L}{\partial q_k} \delta q_k \, dt + \left[\sum_k \frac{\partial L}{\partial \dot{q}_k} \delta q_k\right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) \delta q_k \, dt = 0 \quad \text{--- (4)}$$

But according $\left[\sum_k \frac{\partial L}{\partial \dot{q}_k} \delta q_k\right]_{t_1}^{t_2} = 0$

i.e. $\delta q_k(t_2) = \delta q_k(t_1) = 0$

Then eq^n (4) becomes

$$\int_{t_1}^{t_2} \sum \frac{\partial L}{\partial q_k} \delta q_k \, dt - \int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) \delta q_k \, dt = 0 \quad \text{--- (5)}$$

$$\int_{t_1}^{t_2} \sum_k \left[\frac{\partial L}{\partial q_k} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right)\right] \delta q_k \, dt = 0$$

Since the variation δq_k is arbitrary and independent then.

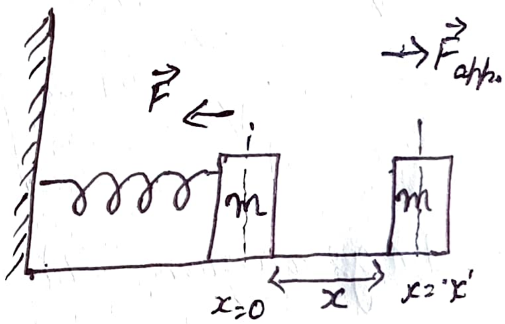
$$\left[\frac{\partial L}{\partial q_k} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right)\right] = 0$$

$$\boxed{\left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k}\right]} = 0 \qquad \text{---} (6)$$

This eq^n is called Lagrange's equation

Linear Harmonic oscillator -

A linear harmonic oscillator consists of particle acted upon by a force whose magnitude changes linearly with distance 'x' from mean position as shown in fig. It oscillates along a straight line about a fixed point known as its mean position. The restoring force acting on linear harmonic oscillator is given by



$$F \propto x$$

[Hook's law]

$$F = -kx \quad \text{——— (1)}$$

K.E. (T) of the particle is given by

$$T = \frac{1}{2} m v^2$$

$$T = \frac{1}{2} m \dot{x}^2 \quad \text{——— (2)}$$

where $v = \dot{x} = \frac{dx}{dt}$

Since restoring force ($\vec{F}$) is a conservative force so it can be represented as

$$F = -\frac{dV}{dx} \Rightarrow dV = -F dx \quad \text{—— (3)}$$

Using (1) in (3)

$$dV = -F dx = (-Kx) dx$$

$$dV = Kx\, dx$$

Integrating both sides

$$\int dV = K \int x\, dx \Rightarrow V = \frac{Kx^2}{2} + c \quad \text{—— (4)}$$

At $x = 0$, $V = 0$ from (4)

$$c = 0$$

$$V = \frac{Kx^2}{2} + 0$$

$$\boxed{V = \frac{kx^2}{2}} \quad \text{—— (5)}$$

Lagrangian (L) of the system is given by

$$L = T - V$$

$$L = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} Kx^2 \quad \text{—— (6)}$$

Partially differentiating (6) w.r.t. $\dot{x}$

$$\frac{dL}{d\dot{x}} = \frac{1}{2} m 2 \dot{x} - 0$$

$$\frac{dL}{d\dot{x}} = m\dot{x} \quad \text{—— (7)}$$

Partially differentiating w.r.t. x

$$\frac{dL}{dx} = 0 - \frac{1}{2}k(2x)$$

$$\frac{dL}{dx} = -kx \quad \text{——— } (8)$$

Lagrangian eq^n of motion is given by

$$\frac{d}{dt}\left(\frac{dL}{d\dot{x}}\right) - \frac{dL}{dx} = 0 \quad \text{——— } (9)$$

Using (7) & (8) in (9)

$$\frac{d}{dt}(m\dot{x}) - (-kx) = 0$$

$$m\ddot{x} + kx = 0$$

~~Using~~ (7) $\ddot{x} + \frac{kx}{m} = 0$

$$\ddot{x} + \omega^2 x = 0 \quad \text{——— } (10)$$

where $\omega^2 = \frac{k}{m} \Rightarrow \omega = \sqrt{\frac{k}{m}}$

$$2\pi\nu = \sqrt{\frac{k}{m}}$$

$$\nu = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

As $T = \frac{1}{\nu}$ so

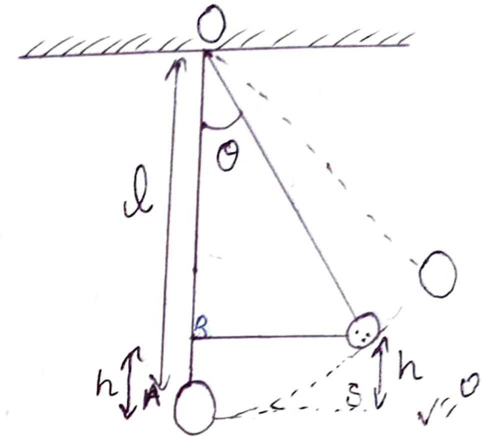
$$\boxed{T = 2\pi\sqrt{\frac{m}{k}}}$$

The sol. of eq^n (10) is

$$x(t) = A\sin(\omega t + \phi)$$

Simple Pendulum :-

A simple pendulum consists of a point mass m suspended at the lower end of a massless & inextensible string of constant length (l) fixed at its upper end to a fixed rigid support



Let θ be angle made by bob at any instant 't' with its mean position A

$$K.E.(T) = \frac{1}{2}mv^2 \quad \text{———} \quad ①$$

$$\text{Angle} = \frac{\text{Arc}}{\text{radius}} \Rightarrow \theta = \frac{s}{l} \Rightarrow s = l\theta \quad \text{———} \quad ②$$

Diff. ② w.r.t. 't'

$$\frac{ds}{dt} = l\frac{d\theta}{dt} \Rightarrow v = l\dot{\theta} \quad \text{———} \quad ③$$

Using ③ in ①

$$T = \frac{1}{2}m(l\dot{\theta})^2$$

$$T = \frac{1}{2}ml^2\dot{\theta}^2 \quad \text{———} \quad ④$$

$$P.E.(V) = mgh = mg(AB) = mg(OA-OB)$$

$V = mg(l - l\cos\theta)$

$= mgl(1-\cos\theta)$ —— (5)

Lagrangian L of the system is

$L = T - V$ —— (6)

Using (4) and (5) in eqn (6)

$L = \frac{1}{2} m l^2 \dot{\theta}^2 - mgl(1-\cos\theta)$ —— (7)

Lagrangian eqn of motion for simple Pendulum

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \quad —— (8)$$

Partially diff. eqn (7) w.r.t. $\dot{\theta}$

$$\frac{\partial L}{\partial \dot{\theta}} = 2 \cdot \frac{1}{2} m l^2 \dot{\theta} - 0$$

$$\frac{\partial L}{\partial \dot{\theta}} = m l^2 \dot{\theta} \quad —— (9)$$

Partially diff. eqn (7) w.r.t. θ

$$\frac{\partial L}{\partial \theta} = 0 - mgl(0 + \sin\theta)$$

$$\frac{\partial L}{\partial \theta} = -mgl\sin\theta \quad —— (10)$$

Using eqn (9) & (10) in (8), then

$$\frac{d}{dt}(ml^2\dot{\theta}) - (-mgl\sin\theta) = 0$$

$$ml^2\ddot{\theta} + mgl\sin\theta = 0$$

Dividing both sides by ml^2

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0$$

For small θ, $\sin\theta \simeq \theta$

$$\boxed{\ddot{\theta} + \frac{g}{l}\theta = 0} \quad \text{——— (a)}$$

This is required lagrangian eq^n of motion for simple pendulum

The general eq^n of S.H.M is

$$\ddot{\theta} + \omega^2\theta = 0 \quad \text{——— (b)}$$

Comparing (a) and (b) we get

$$\omega^2 = \frac{g}{l} \Rightarrow \omega = \sqrt{\frac{g}{l}}$$

$$2\pi\nu = \sqrt{\frac{g}{l}}$$

$$\nu = \frac{1}{2\pi}\sqrt{\frac{g}{l}} \qquad \text{As } T = \frac{1}{\nu}$$

$$T = 2\pi\sqrt{\frac{l}{g}}$$

Using 1 in 8

$$V_2 = m_2 g(l - x_1 - \pi r) \quad \text{— (9)}$$

Total P.E. of system $(V) = V_1 + V_2$

$$V = -m_1 g x_1 - m_2 g(l - x_1 - \pi r)$$

$$= -m g x_1 - m_2 g l + m_2 g x_1 + m_2 g \pi r$$

$$= -g x_1 (m_1 - m_2) - m_2 g(l - \pi r)$$

$$= -g x_1 (m_1 - m_2) + V_0 \quad \text{— (10)}$$

where $V_0 = -m_2 g(l - \pi r)$
$= \text{constant})$

Lagrangian (L) of the system is

$$L = T - V \quad \text{— (11)}$$

Using 6 & 10 in 11

$$L = \frac{1}{2}(m_1 + m_2)\dot{x}_1^2 - g x_1 (m_1 - m_2) + V_0 \quad \text{— (12)}$$

Lagrange's eqⁿ of motion is given by

$$\frac{\partial}{\partial t}\left(\frac{dL}{d\dot{x}_1}\right) - \frac{dL}{dx_1} = 0 \quad \text{— (13)}$$

Partially diff. 12 w.r.t. $\dot{x}_1$

$$\frac{dL}{d\dot{x}_1} = \frac{1}{2}(m_1 + m_2) 2\dot{x}_1 + 0$$

$$\frac{\partial L}{d\dot{x}_1} = \frac{1}{\cancel{2}}(m_1 + m_2)\cancel{2}\dot{x}_1 \quad \text{— (14)}$$

$$\frac{dL}{d\dot{x}_1} = g(m_1 - m_2) \quad \text{— (15)}$$

$$\frac{d}{dt}(m_1+m_2)\dot{x}_1 - g(m_1-m_2) = 0$$

$$(m_1+m_2)\ddot{x}_1 - g(m_1-m_2) = 0$$

$$(m_1+m_2)\ddot{x}_1 = g(m_1-m_2)$$

$$\boxed{\ddot{x}_1 = \frac{g(m_1-m_2)}{m_1+m_2}}$$

Unit - 3

Rotation of a rigid body -

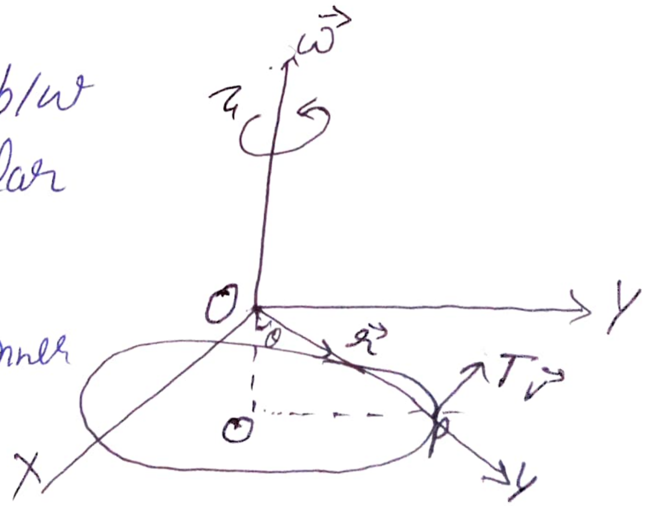
The motion of the particles is confined to the planes perpendicular to the axis of rotation and is called plane rotation or rotation in two dimensions.

The angular velocity $\vec{\omega} = \frac{d\vec{\theta}}{dt}$ is a vector quantity.

In vector rotations,

$$\vec{v} = \vec{r} \times \vec{\omega}$$

Vector relation b/w linear and angular velocities.



In the same manner angular acceleration

$$\alpha = \frac{d\vec{\omega}}{dt}$$

Now,

$$a = \frac{d\vec{v}}{dt} = \frac{d}{dt}(\vec{r} \times \vec{\omega})$$

$$= \left(\frac{d\vec{r}}{dt} \times \vec{\omega}\right) + \left(\vec{r} \times \frac{d\vec{\omega}}{dt}\right)$$

$$= (\vec{v} \times \vec{\omega}) + (\vec{r} \times \vec{\alpha})$$

$$= \vec{a}_r + \vec{a}_t$$

Using law of parallelogram of vectors i.e. along PL in fig.

$$\vec{a}_r = \vec{v} \times \vec{\omega}$$

$$\vec{a}_t = \vec{r} \times \vec{\alpha}$$

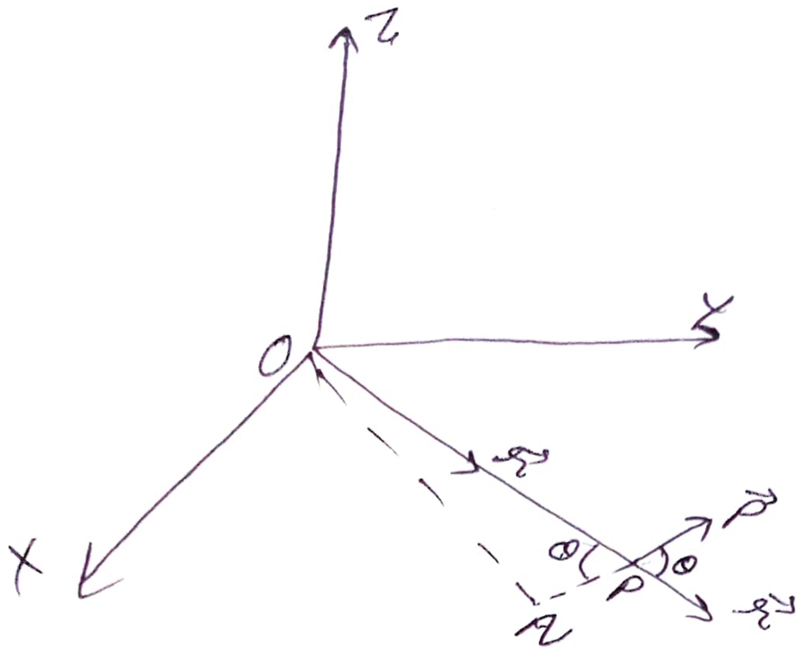
Thus

$a_r = v\omega$ if angle b/w $\vec{v}$ and $\vec{\omega}$ is $90°$

$$\boxed{a_r = r\omega^2}$$

Angular Momentum :-

It may be defined as the moment of the linear momentum of the particle and is given by the product of the linear momentum and $\perp$ distance of its line of action from the axis of rotation. It is denoted by $\vec{L}$ or $\vec{J}$.



Consider a particle P with position vector $OP = \vec{r}$ in the XY plane.

The magnitude of angular momentum $\vec{l}$ is

$$l = ON \times P = r \sin \theta P$$

$$= rP \sin\theta \quad \text{——— (1)}$$

Thus angular momentum of the particle is $\vec{l} = \vec{r} \times \vec{P}$ ——— (2)

It may be noted,

(i) if $\theta = 0°$ or $180°$ i.e. $\vec{P}$ passes through origin and $\vec{l} = 0$

(ii) if $\theta = 90°$ then Angular momentum ($\vec{l}$) is maximum.

Relation b/w Torque and Angular momentum

We know that

$$\vec{l} = \vec{r} \times \vec{p} \quad \text{--- (1)}$$

where, $\vec{l}$ = angular momentum
$\vec{p}$ = linear momentum

Diff. eqn (1) w.r.t. time

$$\frac{d\vec{l}}{dt} = r \times \frac{d\vec{p}}{dt} + \frac{d\vec{r}}{dt} \times \vec{p} \quad \text{--- (2)}$$

But $\frac{d\vec{p}}{dt}$ is the rate of change of momentum $= \vec{F}$, $\frac{d\vec{r}}{dt} = \vec{v}$ and $\vec{p} = m\vec{v}$

Substituting these in eqn (2)

$$\frac{d\vec{l}}{dt} = (\vec{r} \times \vec{F}) + (\vec{v} \times m\vec{v})$$

$$= \vec{r} \times \vec{F} \quad (\because \vec{v} \times \vec{v} = 0)$$

$$= \vec{\tau}$$

Thus $\boxed{\vec{\tau} = \frac{d\vec{l}}{dt}}$

Kinetic Energy of a Rotating Body.

Consider the body to be mass M, made up of a large number of elements of masses m_1, m_2, m_3 .... distant r_1, r_2, r_3 ---- from the axis of rotation XY, having velocities V_1, V_2, V_3 ---.

Then K.E. =

$$\frac{1}{2}m_1V_1^2,\ \frac{1}{2}m_2V_2^2;\ \frac{1}{2}m_3V_3^2$$

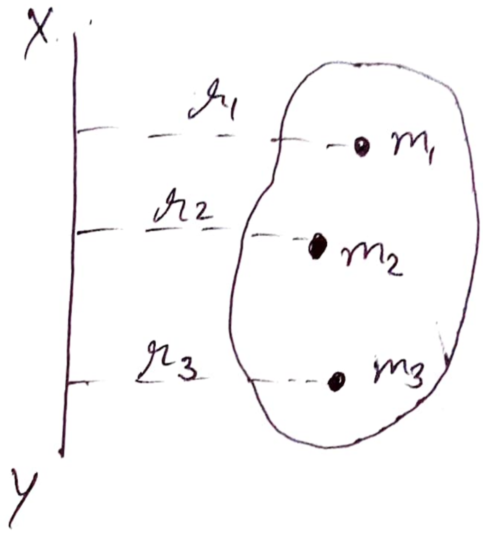


Fig. – K.E. of a rotating body.

K.E. of the whole body = Sum of all elements K.E.

$$K.E. = \frac{1}{2}m_1V_1^2 + \frac{1}{2}m_2V_2^2 + \frac{1}{2}m_3V_3^2 + \cdots$$

$$= \frac{1}{2}m_1(r_1\omega)^2 + \frac{1}{2}m_2(r_2\omega^2) + \frac{1}{2}m_3(r_3\omega)^2 + \cdots$$

$$= \frac{1}{2}m_1r_1\omega^2 + \frac{1}{2}m_2r_2\omega^2 + \frac{1}{2}m_3r_3\omega^2 + \cdots$$

$$= \Sigma\left(\frac{1}{2}mr^2\omega^2\right)$$

$$= \frac{1}{2}\omega^2\,\Sigma(mr^2)$$

$$K.E. = \frac{1}{2}\omega^2\,\Sigma(mr^2).$$

The expression $\Sigma m r^2$ is called moment of inertia of the body and denoted by I its d

Hence, $\boxed{K.E. = \frac{1}{2} I \omega^2}$

Moment of Inertia -

Moment of inertia is the ratio of the torque applied to a body to the resulting angular acceleration.
It's a measure of a body's resistance to having its rotational speed changed by an applied torque.

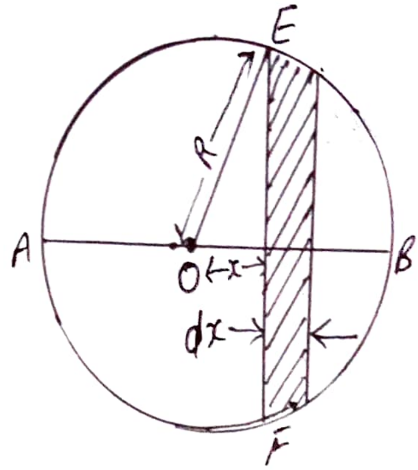
$$I = \Sigma m_i r_i^2$$

It's unit is $kg\, m^2$

Moment of Inertia of a solid Sphere

a) About any diameter
Consider a sphere having mass M and radius R through its centre O. Let AB be any diameter about which

its moment of inertia is to be determined. Consider a thin circular disc EF, of thickness dx and radius y at distance x from centre.



The radius of the disc

$$y = \sqrt{R^2 - x^2}$$

Volume of disc $= \pi(R^2 - x^2)dx$

Volume of the sphere $= \frac{4}{3}\pi R^3$

Mass of disc, $M_1 = \frac{M}{\frac{4}{3}\pi R^3} \times \pi(R^2 - x^2)dx$

$$= \frac{3M(R^2 - x^2)}{4R^3}dx$$

Moment of inertia of the disc about the axis AB is

$$\frac{1}{2}m_1 y^2 = \frac{1}{2}\frac{3M(R^2 - x^2)}{4R^3}dx \times y^2$$

$$= \frac{3M(R^2 - x^2)^2}{8R^3}, \text{ as } y^2 = R^2 - x^2$$

Moment of inertia of the sphere about AB is given by

$$I = 2\int_0^R \frac{3m(R^2 - x^2)^2}{8R^3}dx$$

$$I = \frac{3M}{4R^3} \int_0^R R^4 + x^4 - 2x^2R^2)\,dx$$

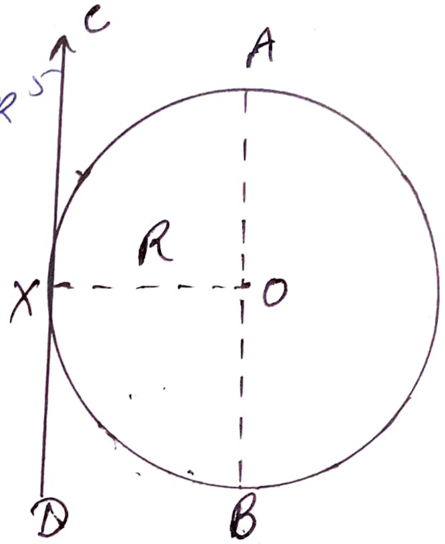
$$= \frac{3M}{4R^3}\left[R^4 x + \frac{x^5}{5} - 2R^2 \times \frac{x^3}{3}\right]_0^R$$

$$= \frac{3M}{4R^3}\left[R^5 + \frac{R^5}{5} - \frac{2R^5}{3}\right]$$

$$= \frac{3M}{4R^3} \times \frac{(15+3-10)R^5}{15}$$

$$= \frac{3M}{4R^3} \times \frac{8R^5}{15}$$

$$I = \frac{2}{5}MR^2$$



b) <u>About a tangent</u> -

The moment of inertia of a sphere about any diameter AB is given by

$$I_{AB} = \frac{2}{5}MR^2$$

The tangent to the sphere at any point is parallel to one of its diameters and is at a distance R from its centre O.

Let CD be the tangent

parallel to the diamenter AB.

The moment of inertia of the inertia of the sphere about the axis CD is given by

$$I_{CD} = I_{AB} + MR^2$$

$$= \frac{2}{5} MR^2 + MR^2 = \frac{7}{5} MR^2$$

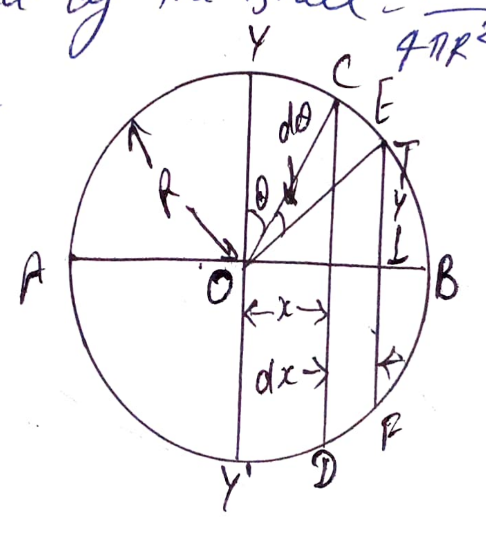
☆ Moment of Inertia of a thin Spherical shell.

a) About any diameter

Consider a section of a thin spherical shell of mass M and radius R through its centre O.

Surface area of the shell $= 4\pi R^2$

Mass per unit surface area of the shell $= \frac{M}{4\pi R^2}$



Let AB be the diameter about which its M.O.I is found. Consider a thin element of shell lying b/w two ∥ planes CD and EF each

$\perp$ to AB and at distances x and $(x + dx)$ from the centre O.

If $\angle YOC = \theta$ and $\angle COE = d\theta$ then

$$y = R\cos\theta \text{ and } x = R\sin\theta$$

Diff. x, we have

$$dx = R\cos\theta\, d\theta = y\, d\theta \quad \text{——} (1)$$

Surface area of the ring = Circumference × width

$$= 2\pi y \times CE = 2\pi y \times R\, d\theta$$

$$= 2\pi R y\, d\theta$$

$$= 2\pi R\, dx \qquad [y\, d\theta = dx]$$

Mass of slice (ring) $= \dfrac{M}{4\pi R^2} \cdot 2\pi R\, dx$

$$= \frac{M}{2R} dx$$

Moment of inertia of slice about AB.

$$= (\text{its mass}) \times (\text{radius})^2$$

$$= \left(\frac{M}{2R} dx\right) y^2 = \frac{M\, dx}{2R}(R^2 - x^2)$$

$\therefore$ Moment of inertia of the shell

$$= 2\int_0^R \frac{M}{2R}(R^2 - x^2)\, dx$$

$$= \frac{M}{R}\left[R^2 x - \frac{x^3}{3}\right]_0^R = \frac{M}{R}\left[R^3 - \frac{R^3}{3}\right]$$

$$\boxed{I = \frac{2}{3} M R^2}$$

b) About a tangent.

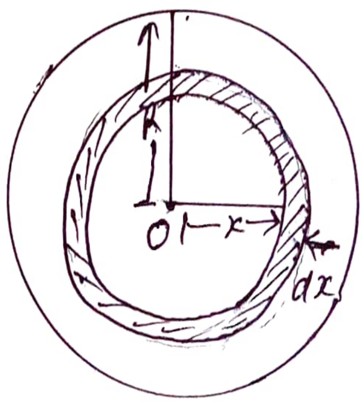
Moment of inertia of the shell about any tangent, parallel to the AB - axis can be calculated by principle of ∥ axes and is given by

$$I_T = I + MR^2 = \frac{2}{3}MR^2 + MR^2$$

$$\boxed{I_T = \frac{5}{3}MR^3}$$

Moment of inertia of solid sphere -

Consider the solid sphere to be divided into a large no. of concentric shells of infinitesi-mal thickness. Let a shell of small thickness dx be at a distance x from the centre and mass of sphere be M.



Volume of sphere $= \frac{4}{3}\pi R^3$

Mass ~~per~~ per unit volume of sphere

$$= \frac{M}{\frac{4}{3}\pi R^3} = \frac{3M}{4\pi R^3}$$

Volume of the shell $= 4\pi x^2 dx$

Mass of the shell $= \frac{3M}{4\pi R^3} \times 4\pi x^2 dx$

$$= \frac{3M}{R^3} x^2 dx$$

Moment of inertia of the shell about any diameter

$$= \frac{2}{3} (\text{mass of shell}) \times (\text{radius})^2$$

$$= \frac{2}{3} \left(\frac{3M}{R^3} x^2 dx \right) \times x^2$$

$$= \frac{2M}{R^3} x^4 dx$$

The moment of inertia of the sphere about a diameter.

$$= \frac{2M}{R^3} \int_0^R x^4 dx = \frac{2M}{R^3} \left[\frac{x^5}{5} \right]_0^R$$

$$= \frac{2M}{R^3} \left(\frac{R^5}{5} \right) = \frac{2}{5} MR^2$$

$$\boxed{I = \frac{2}{5} MR^2}$$

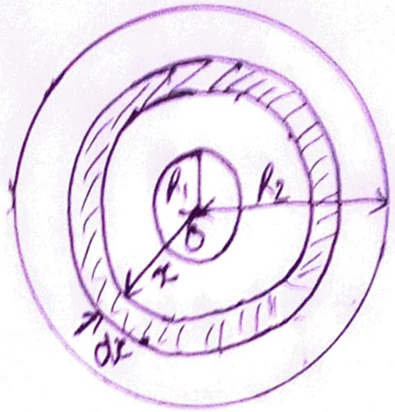
4. Moment of inertia of a Hallow sphere

Let M be the mass of a hallow sphere, whose inner and outer radii are R_1 and R_2 respectively.

Volume of this sphere $= \frac{4}{3}\pi R_2^3 - \frac{4}{3}\pi R_1^3$

$$V = \frac{4}{3}\pi(R_2^3 - R_1^3)$$

Mass per unit volume of the sphere



$$\rho = \frac{M}{\frac{4}{3}\pi(R_2^3 - R_1^3)}$$

$$= \frac{3M}{4\pi(R_2^3 - R_1^3)}$$

Imagine the hollow sphere to be divided into a no. of thin spherical shell of infinitesimal thickness.

Volume of shell $= 4\pi x^2 dx$

Mass of shell $= 4x^2 dx \times \rho$

$$= 4\pi x^2 dx \times \frac{3M}{4\pi(R_2^3 - R_1^3)}$$

$$= \frac{3M}{(R_2^3 - R_1^3)} x^2 dx$$

Moment of inertia of the shell about a diameter

$$= \frac{2}{3}\left[\frac{3M}{(R_2^3 - R_1^3)} x^2 dx\right] \times x^2 = \frac{2M}{(R_2^3 - R_1^3)} x^4 dx$$

Hence the moment of inertia of whole hollow sphere about its diameter is given by

$$I = \int_{R_1}^{R_2} \frac{2M}{(R_2^3 - R_1^3)} x^4 dx$$

$$I = \frac{2M}{R_2^3 - R_1^3} \int_{R_1}^{R_2} x^4 dx$$

$$= \frac{2M}{R_2^3 - R_1^3} \left[\frac{x^5}{5}\right]_{R_1}^{R_2}$$

$$= \frac{2}{5} M \left[\frac{R_2^5 - R_1^5}{R_2^3 - R_1^3}\right]$$

If $R_1 = 0$, $R_2 = R$, we have for a solid sphere.

$$\left[I_s = \frac{2}{5} M \left(\frac{R^5}{R^3}\right) = \frac{2}{5} MR^2 \right]$$

Moment of inertia of a solid cylinder -

a) About its own axis -

We have a solid cylinder of Radius R and Mass M. We find the moment of inertia of the cylinder about the axis AB of the cylinder.

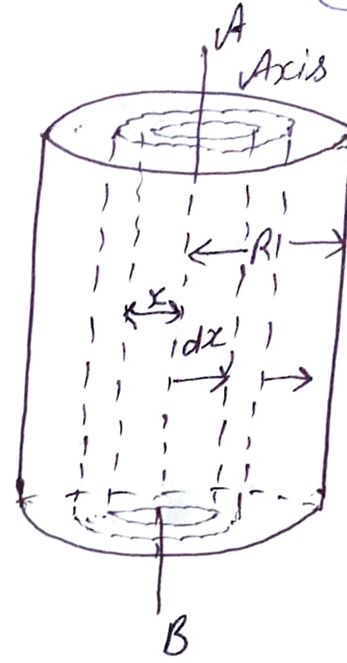
Consider a cylinder of radius x and thickness dx. All particles of this cylinder are at same distance x from the axis of rotation AB. The volume of elementary cylinder

$2\pi x\, dx \times l$

Volume of the given cylinder

$= \pi R^2 l$

Mass per unit volume $= \dfrac{M}{\pi R^2 l}$



$\therefore$ Mass of the elementary cylinder

$$= \frac{M}{\pi R^2 l} \times 2\pi x\, dx\, l$$

$$= \frac{2Mx\,dx}{R^2}$$

Moment of Inertia of the elementary cylinder about the given axis

$$= \frac{2M}{R^2} x \cdot dx \cdot x^2 = \frac{2M}{R^2} x^3 \cdot dx$$

Hence moment of inertia of whole cylinder about AB is

$$I = \int_0^R \frac{2M}{R^2} x^3 \cdot dx = \frac{2M}{R^2}\left[\frac{x^4}{4}\right]_0^R$$

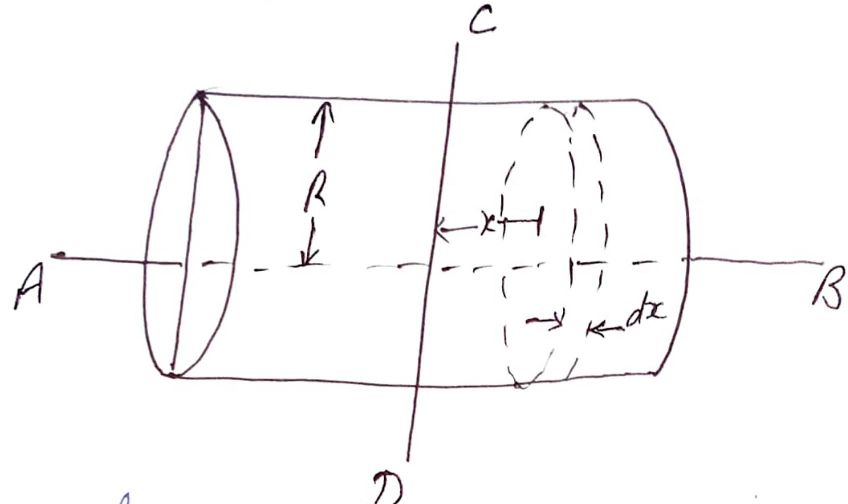
$$= \frac{2M}{R^2} \cdot \frac{R^4}{4}$$

$$\boxed{I = \frac{1}{2} MR^2}$$

b) About an axis passing through its centre of mass but ⊥ to its length

Let CD be axis of rotation of given cylinder of mass M and length l, so that CD is ⊥ to axis AB of the cylinder and passes its center of mass G.

Consider an disc of disc thickness dx and distant x from CD.



Volume of the disc $= \pi R^2 dx$

" " " cylinder $= \pi R^2 l$

Mass per unit volume $= \dfrac{M}{\pi R^2 l}$

$\therefore$ Mass of disc $= \dfrac{M}{\pi R^2 l} \cdot \pi R^2 dx$

$= \dfrac{M}{l} dx$

Moment of inertia of the disc about a diameter, which is parallel the axis

$$\frac{M}{l} dx \cdot \frac{R^2}{4}$$

Thus moment of inertia of disc about CD by the principle of the parallel axes is

$$\frac{M}{l} dx \frac{R^2}{4} + \frac{M}{l} dx \cdot x^2$$

Moment of inertia of whole cylinder about CD is

$$I = \int_{-l/2}^{l/2} \left(\frac{MR^2}{4l} dx + \frac{M}{l} x^2 \cdot dx \right)$$

$$= \frac{MR^2}{4l} \int_{-l/2}^{l/2} dx + \frac{M}{l} \int_{-l/2}^{l/2} x^2 dx$$

$$= \frac{MR^2}{4l} \cdot l + \frac{M}{l} \left[\frac{x^3}{3} \right]_{-l/2}^{l/2}$$

$$= \frac{MR^2}{4} + \frac{Ml^2}{12}$$

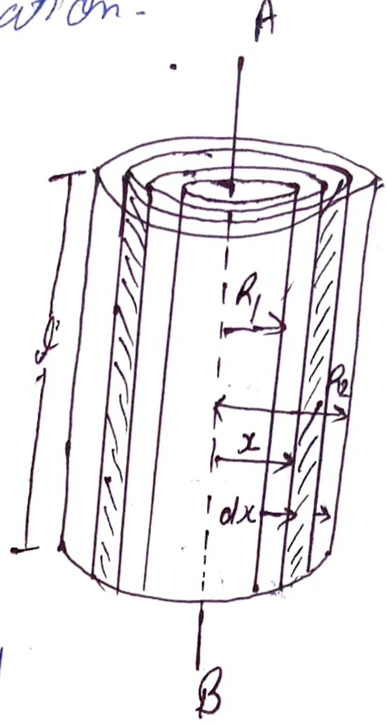
$$I = M \left[\frac{R^2}{4} + \frac{l^2}{12} \right]$$

* Moment of inertia of a hollow circular cylinder.

a) About its own axis -

Let AB be axis of a hollow cylinder

of length l, mass M and internal and external radii R_1 and R_2. AB is also axis of rotation.



$2\pi x dx \times l$

Volume of the given cylider

$$= \pi (R_2^2 - R_1^2) l$$

Mass per unit volume of the cylider $= \dfrac{M}{\pi (R_2^2 - R_1^2) l}$

Mass of the elementary cylinder

$$= \frac{M}{\pi (R_2^2 - R_1^2) l} \times 2\pi x dx \times l$$

$$= \frac{2Mx\,dx}{(R_2^2 - R_1^2)}$$

Moment of interia of the elemantary cylinder $= \dfrac{2Mx\,dx}{(R_2^2 - R_1^2)} \times x^2$

Hence moment of intertia of the whole cylider about AB

$$\therefore \quad I = \int_{R_1}^{R_2} \frac{2Mx^3\,dx}{(R_2^2 - R_1^2)} = \frac{2M}{(R_2^2 - R_1^2)} \int_{R_1}^{R_2} x^3\,dx$$

$$= \frac{2M}{(R_2^2 - R_1^2)} \left[\frac{x^4}{4}\right]_{R_1}^{R_2}$$

$$= \frac{2M}{R_2^2 - R_1^2} \times \frac{R_2^4 - R_1^4}{4} = \frac{M(R_1^2 + R_2^2)}{2} \quad \text{--- (3)}$$

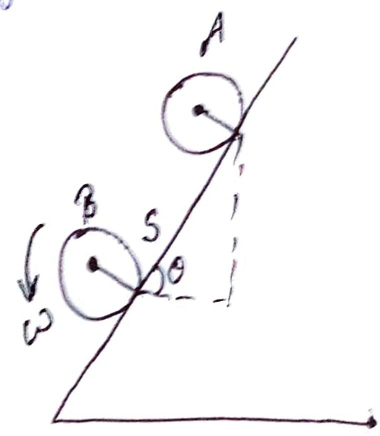
For a solid cylider of radius R having no inner hole $R_1 = 0$ and $R_2 = R$,

$$I = \frac{MR^2}{2}$$

Acceleration of a body rolling down on an inclined plane -

The acc. of a body rolling down an inclined plane is given by the formula $a = g \sin \theta_1 + K_2 R_2$. In this formula, a is ~~the~~ the acc., g is the due to gravity, I is the moment of inertia, M is the mass, and R is the radius.

Here are some other things.



Gain in K.E of translation

$$= \frac{1}{2} mv^2$$

Gain in K.E of rotation

$$= \frac{1}{2} I\omega^2$$

$$= \frac{1}{2} Mk^2 \frac{v^2}{R^2}$$

$$= \frac{1}{2} mv^2 \left(1 + \frac{k^2}{R^2}\right)$$

Total Gain in K.E. $= \frac{1}{2} mv^2 + \frac{1}{2} mv^2 \frac{k^2}{R^2}$

Vertical height descended by the body

$$h = S \sin \theta$$

$\therefore$ Loss in P.E. of the body $= mgh = mg S \sin\theta$

Gain in K.E. = Loss in P.E.

$$\frac{1}{2} mv^2 \left[1 + \frac{k^2}{R^2}\right] = Mg\, S \sin\theta$$

$$V^2 = 2\left[\frac{g \sin\theta}{1 + k^2/R^2}\right] S \quad \text{——— (1)}$$

Thus

$$V = \sqrt{\frac{2gh}{\left(1 + \frac{k^2}{R^2}\right)}}$$

Since the body starts from rest therefore its 'a' down the inclined plane is given by

$$V^2 = 2as \quad \text{——— (2)}$$

Compairing (1) and (2), we get

$$\boxed{a = \frac{g \sin\theta}{1 + \frac{k^2}{R^2}}}$$

If t be the time of descent, then

$$S = ut + \frac{1}{2}at^2 = \frac{1}{2}at^2$$

$$\frac{h}{\sin\theta} = \frac{1}{2}\left[\frac{g \sin\theta}{1 + \frac{k^2}{R^2}}\right] t^2$$

$$t = \frac{1}{\sin\theta}\sqrt{\frac{2h}{g}\left(1 + \frac{k^2}{R^2}\right)}$$

Motion involving both translational and rotation -

We shall show that L_z the z component of the angular momentum of the body can be written as the sum of two terms, L_z is the angular momentum $I_o \omega$ due to rotation of the body about its center of mass.

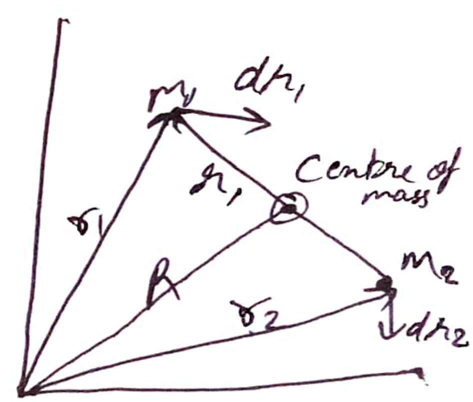
$$L_z = I_o \omega + (\bar{R} \times M \bar{V})_z$$

where R is the position vector of the centre of mass and

$$V = \dot{R}$$

Consider that the body is made up of N particles with masses m_j $(j = 1, 2, —)$
The angular momentum of the body can be written as

$$L = \Sigma (r_j \times m_j \dot{r}_j)$$



The centre of mass of the body

$$R = \frac{\Sigma m_j r_j}{M}$$

Introducing the centre of mass coordinate

$$r_j = R + r_j'$$

$$L = \sum (r_j \times m_j \dot{r}_j)$$

$$= \sum (R + r_j') \times m_j (\dot{R} + \dot{r}_j')$$

$$= R \times \sum m_j \dot{R} + \sum m_j \dot{r}_j' \times \dot{R} +$$

$$R \times \sum m_j \dot{r}_j' + \sum m_j r_j' \times \dot{r}_j'$$

$$\sum m_i r_j' = \sum m_j (r_j - R)$$

$$= \sum m_i r_j - MR$$

$$= 0$$

The first term is

$$R \times \sum m_j \dot{R} = R \times M \dot{R}$$

$$= R \times MV$$

where $V = \dot{R}$ is the velocity of the center of mass w.r.t. inertial system. The expression for L then becomes

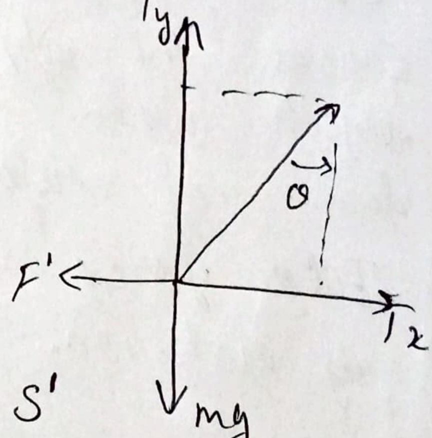
$$L = R \times MV + \sum r_j' \times m_j \dot{r}_j'$$

Unit - 4

Non- inertial frames of reference -

A frame of reference in which Newton's law of motion are not valid is known as Non - Inertial frame of reference.

A non intertial frame of reference is an accelerated frame of referame



Fictitions force -

Fictitions force, also known as pseudo forces, are forces that appear to act on an object when viewed from a non-inertial frame of reference. They are used to explain motion in non- inertial frames, such as rotating or ac-

-celerated frames, and to maintain the validity of Newton's second law of motion.

Some examples of fictitious forces-

- Centrifugal force-
  Tends to throw an object off when the object is rotating in a non-inertial frame of reference.
- Coriolis force-
  Causes the apparent deflection of moving objects when viewed in a rotating frame of reference.
- Euler force- caused by a variable rate of rotation.

Uniformaly Rotating frame

A uniformly rotating frame is a non-inertial frame of reference in which a particle moves in a circle at a constant velocity around the z-axis of an inertial frame.

②

$$S \Rightarrow \quad \vec{r} = (x\hat{i} + y\hat{j} + z\hat{k})$$

$$\vec{r'} = (x'\hat{i'} + y'\hat{j'} + z'\hat{k'})$$

$$\frac{d\vec{r}}{dt} = \left(x'\frac{d\hat{i'}}{dt} + \hat{i}\frac{dx'}{dt} + y'\frac{d\hat{j}}{dt} + \hat{j}\frac{dy'}{dt} + z'\frac{d\hat{k'}}{dt} + \hat{k}\frac{dz'}{dt} \right.$$

$$\frac{d\vec{r}}{dt} \text{ or } \dot{\vec{r}} = \hat{i}'\dot{x}' + \hat{j}\dot{y} + \hat{k}\dot{z}' + \omega \times (x'\hat{i} + \cancel{y'_z}\ \hat{j}y' + z\hat{k}')$$

$$= i'\dot{x}' + \hat{j}'\dot{y}' + \hat{k}\dot{z}' + (\omega \times \vec{r})$$

$$\vec{v} = v' + (\omega \times \vec{r})$$

$$\frac{d\vec{v}}{dt} = \frac{d'v'}{dt} + \vec{\omega} \times \vec{v}$$

$$= \frac{d'}{dt}(v' + (\omega \times \vec{r}) + \vec{\omega} \times (v' + \vec{\omega} \times \vec{r})$$

$$= \frac{d'v}{dt} + \omega \times \frac{d'\vec{r}}{dt} + \vec{\omega} \times v' + \vec{\omega} \times (\vec{\omega} \times \vec{r})$$

$$= \frac{d'v'}{dt} + \omega \times V' + \vec{\omega} \times V' + \vec{\omega} \times (\vec{\omega} \times \vec{r})$$

$$= \frac{d'v'}{dt} + 2(\omega \times v') + \vec{\omega} \times (\vec{\omega} \times \vec{r})$$

$$\vec{a} = \vec{a}' + 2(\omega \times v') + \vec{\omega} \times (\vec{\omega} \times \vec{r})$$

$$F = F' + 2m(\omega \times v') + \vec{\omega} \times (\vec{\omega} \times \vec{r})$$

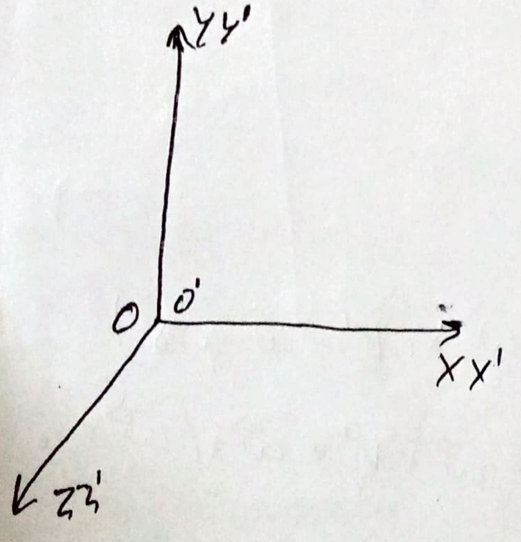
$$ma = ma' - ma$$

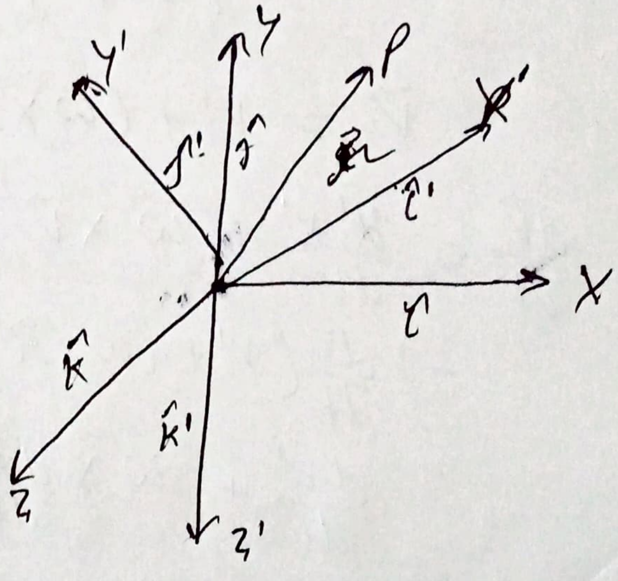
① Centrifugal force $= -2m(\omega \times v')$

② Coriolis force $= -\vec{\omega} \times (\vec{\omega} \times \vec{R})$

Law of Physics in rotating co-ordinate systems:-

Let us consider two co-ordinate systems unprimed $O(x, y, z)$ and primed $O'(x', y', z')$ having common origin as shown in figure.





$$V_s = V_R + x'\frac{d\hat{i}'}{dt} + y'\frac{d\hat{j}'}{dt} + z'\frac{d\hat{k}'}{dt}$$

Consider a radius vector $\vec{R}$ rotates about axis with an angular velocity as shown in fig.

$$d\theta = \frac{\text{Arc of length (AB)}}{\text{Radius (OA)}}$$

Centrifugal force -

Centrifugal force is a fictitious force that appear to act on objects when viewed in a rotating frame of reference. It's useful in many appli-cations, including.

Centrifuges -
Centrifugal force is used in centrifuges to separate particles suspended in a fluid by spinning at high speeds.

Road construction -
Centrifugal force is used to improve friction on curves and avoid skidding.

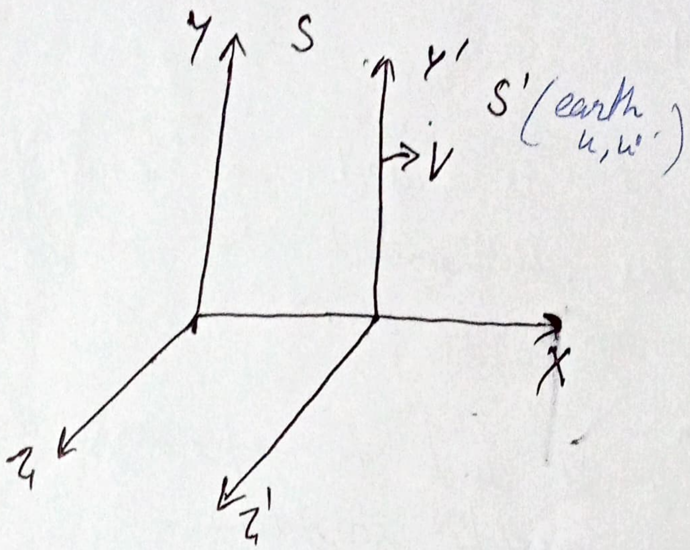
Washing machines and car turns -
Centrifugal force is used to analyze the behaviour of rotating devices like centrifugal pumps, governors and clutches.

Centrifugal force is always directed away from the axis of rotation. It can be increased by!

- Increasing the speed of rotation
- Increasing the mass of the body
- Decreasing the radius, which is the distance of the body from the center of the curve.

Michelson - Morley Experiment -

<u>Introduction</u> - Here we have an inertial frame S and S' which are having (V, V') velocity.



$$C = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$$

μ_0 = Absolute permeability of free space

ε_0 = Absolute electectrial permittivity of free space

$C = 2.99 \times 10^8 m/s = 3 \times 10^8 m/s$

$u' = u - V$

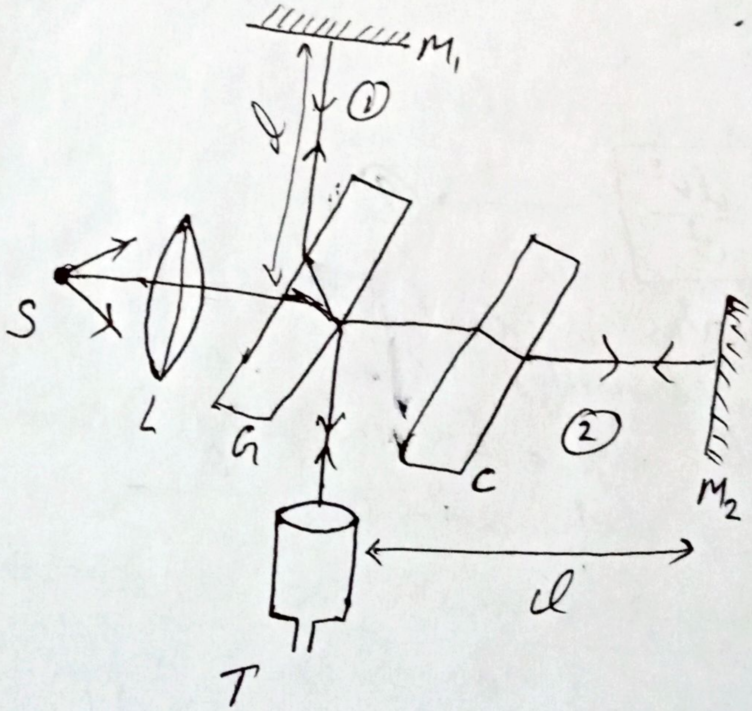
$u' = c - V$

$u' = c + V$

Michlesons Morley Experiment -

Michlesons morley performed an historic experiment in which down the contraversy regarding existence of device will used a very sensitive device ~~eor~~ easier deligned by Michleson in 1837 and used a very sensitive device easier designed, they tried to check the variation known as Michleson's Interfermeter and they tried to check the variation in velocity of light measured w.r.t. earth moving through ether - If variation in velocity is observed then it will uniform

Let t_1 time taken by light



S = source of monochromatic light

G = glass plate

C = Compensating plate

M_1, M_2 = Highly polished plane Glass mirrors

T = Telescope

L = Converging lens

$$t_1 = \frac{l}{c-v} + \frac{l}{c+v}, \quad t_1 = \left[\frac{c+v+c-v}{(c-v)(c+v)}\right] l$$

$$t_1 = \frac{2lc}{c^2-v^2} = \frac{2lc}{c^2\left(1-\frac{v^2}{c^2}\right)} = \frac{2l}{c}\left(1-\frac{v^2}{c^2}\right)^{-1} \quad \text{--- (1)}$$

$$t_1 = \frac{2l}{c}\left(1 + \frac{v^2}{c^2}\right) \quad \text{(1)}$$

Let t_2 time taken by beam (1) in going from G to M_1

$$t_2 = \frac{l}{\sqrt{c^2 - v^2}} + \frac{l}{\sqrt{c^2 - v^2}} = \frac{2l}{\sqrt{c^2 - v^2}} = \frac{2l}{c\sqrt{1 - \frac{v^2}{c^2}}}$$

$$t_2 = \frac{2l}{c}\left(1 - \frac{v^2}{c^2}\right)^{-1/2} = \frac{2l}{c}\left(1 + \frac{v^2}{2c^2}\right) \quad \text{(2)}$$

Time diff. b/w beam (1) & (2) is

$$\Delta t = t_1 - t_2 = \frac{2l}{c}\left(1 + \frac{v^2}{c^2}\right) - \frac{2l}{c}\left(1 + \frac{v^2}{2c^2}\right)$$

$$\boxed{\Delta t = \frac{l v^2}{c^3}} \quad \text{(3)}$$

Path difference $= c \times \Delta t$

$$= c \times \frac{l v^2}{c^3}$$

$$\boxed{\Delta x = \frac{l v^2}{c^2}} \longrightarrow \text{(4)}$$

When whole apparatus is rotated through $90°$ reflected and transmitted beam are interchange and rotating path diff. b/w interfering beams becomes

$$\frac{l v^2}{c^2} + \frac{l v^2}{c^2} = \frac{2 l v^2}{c^2}$$

$$\text{Path diff.} = \frac{2 l v^2}{c^2} = n\lambda \Rightarrow n = \frac{2 l v^2}{c^2 \lambda}$$

Conclusions –

(1) Either the ether does not exists . it can't be detected by any experimental means.

(2) The velocity of light in free space is same in all directions and does not depend on velocity of source of light or observer.

Explanations of -ve result-

1. Ether drag - Michelson himself supported -ve result by stating that earth dragged the ether along with it so there is no relative motion b/w earth and ether and hence no shift is observed

☆ Postulates of Special theory of relativity-

Postulate-I -
(Principle of equivalence of all inertial frames)

"The laws of physical phenomenon are exactly same in all inertial frames moving at constant velocities relative to each other".

Postulate - II (Principle of invarience of velocity of light)

"The velocity of light in space has same value for all observers in all directions and is independent of velocity of source and observer.

☆ Frequency and wave number

Frequency and wave no. are both properties of waves but they are diff. in nature:

Frequency –

The number of wave cycles that pass through a point in on sec. Frequency is measured in hertz (Hz) which is equal to one occurrence per second.
It is denoted by (ν).

Wave number –

The no. of wavelength per unit distance
Wave no. is measured in cycles per unit distance or radians per unit distance.
Wave no. and frequency are related to each other and their product is equal to the reciprocal of wave speed.
There are two types of wave number.

- Linear wave no. –
  Based on the no. of cycles per unit distance.
- Angular wave no. –
  Based on radians per unit distance.

Coriolis force -

It is a fictitious force which appears to be acting on a particle moving with velocity $\vec{v}$ with respect to observer connected to rotating frame.

We obtain relation connecting inertial acce. of particle of mass m at P and its acc. relative to rotating frame. we have,

$$\left(\frac{dV_s}{dt}\right)_s = \left(\frac{dV_s}{dt}\right)_r + \omega \times V_s \quad \text{——— (1)}$$

Replacing V_s on right side we get

$$V_s = V_r + \omega \times r$$

$$\left(\frac{dV_s}{dt}\right)_s = \left(\frac{dV_r}{dt}\right)_r + \left(\frac{d(\omega \times r)}{dt}\right)_r + \omega \times V_r + \omega \times (\omega \times r)$$

$$= \left(\frac{dV_r}{dt}\right)_r + \left(\frac{d\omega \times r}{dt}\right)_r + \omega \times V_r + \omega \times V_r + \omega \times (\omega \times r)$$

when angular velocity is constant, $\left(\frac{d\omega}{dt}\right) = 0$

Then factor $\left(dV_s/dt\right)_s$ is inertial acc. $a_s = \left(\frac{dV_r}{dt}\right)_r$ is acceleration a_r.

$$\therefore a_s = a_r + 2(\omega \times V_r) + \omega \times (\omega \times r) \quad \text{——— (2)}$$

equation of motion in inertial system

$$F = ma_s \quad \text{——— (3)}$$

Multiplying by m in eq^n (2)

$$ma_s = ma_r + 2m(\omega \times V_r) + m\omega \times (\omega \times r)$$

Now from eq^n (3)

$$F = -2m(\omega \times V_r) + m\omega \times (\omega \times r) = ma_r$$

To observer rotating system, appears as particle's moving under effective force

$$F_{eff} = F - 2m(\omega \times V_r) - m\omega \times (\omega \times r)$$

IIIrd term $-m\omega \times (\omega \times r)$ is called centrifugal force

$$[m\omega \times (\omega \times r)] = mr\omega^2 \sin\theta$$

θ is the angle b/w ω and r

Second term : $2m(\omega \times V_r)$.

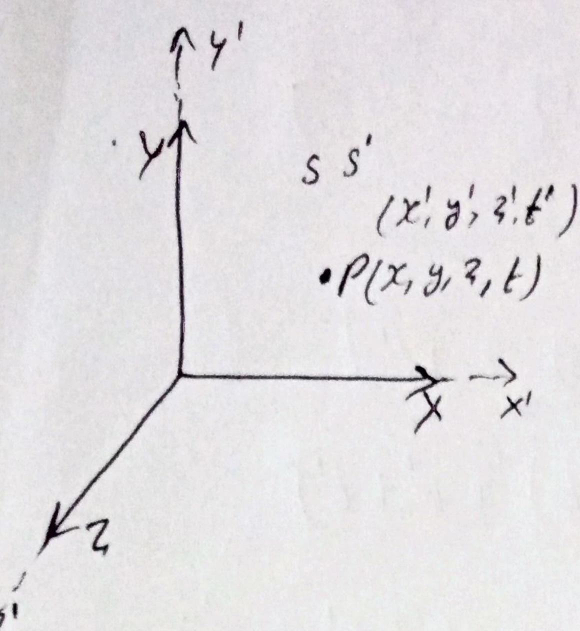
Corious force

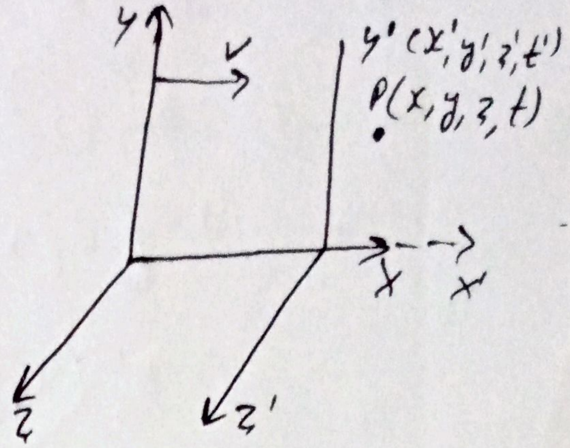
Directly proportional to V_r and will dis-appear when there is no motion.

Application.

- It is taken into account to compute accurately trajectories of long range particle & missible.
- Spinning motion of earth is that causes equatorial bulge.

Lorentz Transformation





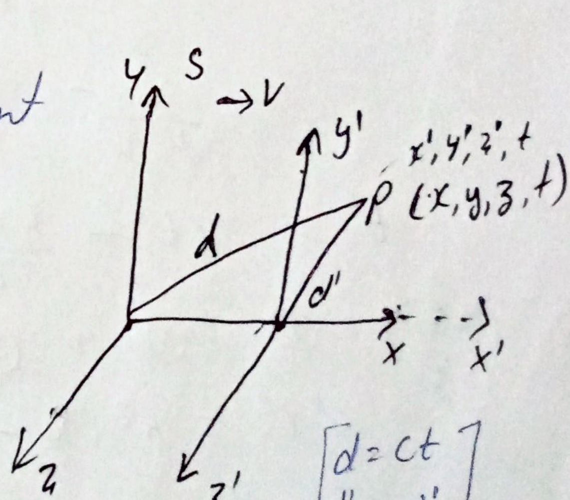
"The set of equations which relates the space-time co-ordinates of two frame of references having relative motion b/w them."

Lorentz transformation eq.^{n}

The eq.^{n} of spherical wavefront is given by

$$x^2 + y^2 + z^2 = d^2$$

$$x^2 + y^2 + z^2 = c^2t^2 \quad \text{—— (1)}$$



$$\left[\begin{array}{l} d = ct \\ d' = ct' \end{array}\right]$$

similarly

$$x'^2 + y'^2 + z'^2 = c^2t'^2 \quad \text{—— (2)}$$

Adding (1) & (2)

$$x^2 + y^2 + z^2 + x'^2 + y'^2 + z'^2 = c^2t^2 + c^2t'^2$$

$$x^2 + y^2 + z^2 - c^2t^2 = x'^2 + y'^2 + z'^2 - c^2t'^2 \qquad \left[\because \begin{array}{l} y' = y \\ z' = z \end{array}\right]$$

$$x^2 - c^2t^2 = x'^2 - c^2t'^2 \quad \text{—— (3)}$$

Relation b/w x & x' is given by

$$x = k(x - vt) \quad \text{—— (4)}$$

$$x = k'(x' + vt') \quad \text{—— (5)}$$

Using (4) in (5)

$$x = k'[k(x - vt) + vt']$$

$$= [kk'(x - vt) + k'vt']$$

$$= kk'(x - vt) + k'vt'$$

$$x - \cancel{kk'}(x - vt) = k'vt'$$

$$t' = \frac{x - kk'(x - vt)}{k'v}$$

$$= \frac{x}{k'v} - \frac{kk'x}{k'v} + \frac{kk'vt}{k'v}$$

$$t' = kt - \frac{kx}{v} + \frac{x}{k'v} \quad \text{—— (5)}$$

Using (4) & (5) in (3)

$$x^2 - c^2t^2 = k^2(x - vt)^2 - c^2\left[k\left\{t - \frac{x}{v}\left(1 - \frac{1}{kk'}\right)\right\}\right)^2$$

$$x^2 - c^2t^2 = k^2x^2 + k^2v^2t^2 - k^2 2xvt - c^2k^2t^2 - c^2k^2\frac{x^2}{v^2}\left(1 - \frac{1}{kk'}\right)$$

$$+ 2\frac{txc^2}{v}k^2\left(1 - \frac{1}{kk'}\right)\Big]$$

$$x^2 - c^2t^2 - k^2x^2 - k^2v^2 + k^2 2xvt + c^2k^2t^2 + \frac{c^2k^2x^2}{v^2}\left(1 - \frac{1}{kk'}\right)^2$$

$$- 2\frac{txc^2k^2}{v}\left(1 - \frac{1}{kk'}\right) = 0$$

$$x^2\left\{1 - k^2 + \frac{c^2k^2}{v^2}\left(1 - \frac{1}{kk'}\right)^2\right. + 2xt\left[k^2v - \frac{c^2k^2}{v}\left(1 - \frac{1}{kk'}\right)\right)$$

$$-t^2\left(c^2+k^2v^2-c^2k^2\right)=0$$

Equating the constant term to zero. we get

$$c^2+k^2v^2-c^2k^2=0$$

$$c^2-k^2(c^2-v^2)=0$$

$$k^2(c^2-v^2)=c^2$$

$$k^2=\frac{c^2}{c^2-v^2}=\frac{c^2}{c^2\left(1-\frac{v^2}{c^2}\right)}=\frac{1}{1-\frac{v^2}{c^2}}$$

$$k=\sqrt{\frac{1}{1-\frac{v^2}{c^2}}}=\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \quad — (6)$$

Equating co-efficent of 2xt to zero

$$k^2v-\frac{c^2k^2}{v}\left(1-\frac{1}{kk'}\right)=0 \quad — (7)$$

Now from (6) & (7)

$$k'=\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \quad — (8)$$

Using (6) in (*), we get   Here $x'=k(x-vt)$ — *

and $t'=k\left[t-\frac{x}{v}\left(1-\frac{1}{kk'}\right)\right]$ **

$$x'=\frac{x-vt}{\sqrt{1-\frac{v^2}{c^2}}} \quad — (9)$$

Using (6) & (8) in (**) then

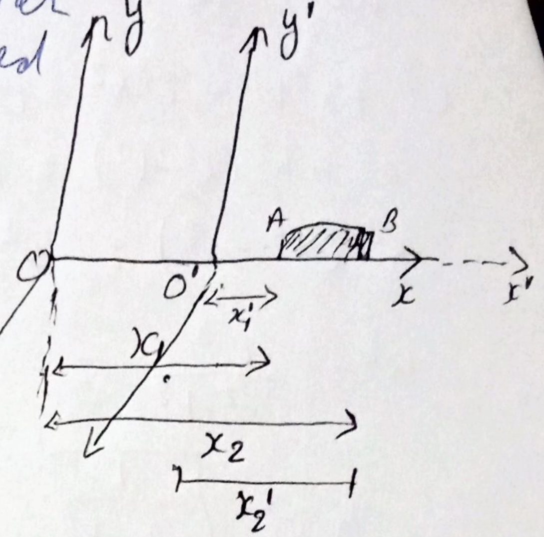
$$t'=\frac{t-\frac{xv}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}$$

The complete set of lorentz transformation eq.^{n} are

$x'=\frac{x-vt}{\sqrt{1-\frac{v^2}{c^2}}}$	$x=\frac{x'+vt'}{\sqrt{1-\frac{v^2}{c^2}}}$
$y'=y$	$y=y'$
$z'=z$	$z=z'$
$t'=\frac{t-\frac{xv}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}$	$t=\frac{t'+\frac{x'v}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}$

Lorentz contraction -

It is the phenomenon of a moving object's length is measured to be shorter than its proper length and measured by ob. own rest frame.



$$L_0 = x_2 - x_1 \quad \text{—①}$$

$$L = x_2' - x_1' \quad \text{—②}$$

Acc. to Lorentz transformation Eq^n

$$x = \frac{x' + vt'}{\sqrt{1 - v^2/c^2}}$$

$$x_1 = \frac{x_1' + vt'}{\sqrt{1 - v^2/c^2}} \; ; \quad x_2 = \frac{x_2' + vt'}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Now

$$L_0 = \frac{x_2' - x_1'}{\sqrt{1 - v^2/c^2}} = \frac{L}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$\boxed{L = L_0 \sqrt{1 - \frac{v^2}{c^2}}}$$

$$L << L_0$$

Ques - let $L_0 = 100$ cm. and $v = \frac{c}{2}$ then find L ?

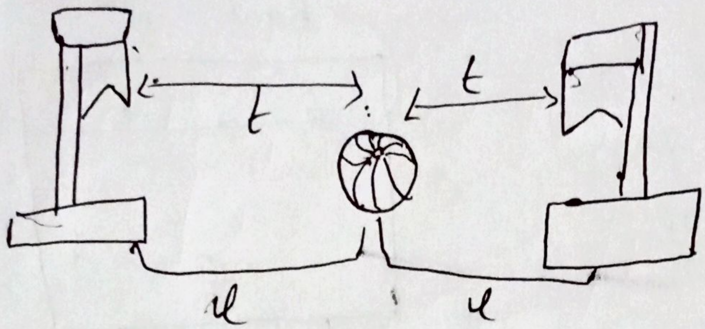
$$L_0 = \frac{L}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{L}{\sqrt{1 - \frac{1}{4}}}$$

$$L_0 = \frac{2L}{\sqrt{3}} \Rightarrow 100 = \frac{2L}{\sqrt{3}} \quad [\because L_0 = 100]$$

$$L = \frac{100 \times 1.73}{2} = 86.4 \text{ cm.}$$

Relativity of Simultaneity -

Simulataneity - Simultaneity is the relation b/w two events assumed to be happening at the same time in a given frame of reference.



$c = 3 \times 10^8 \, m/s$

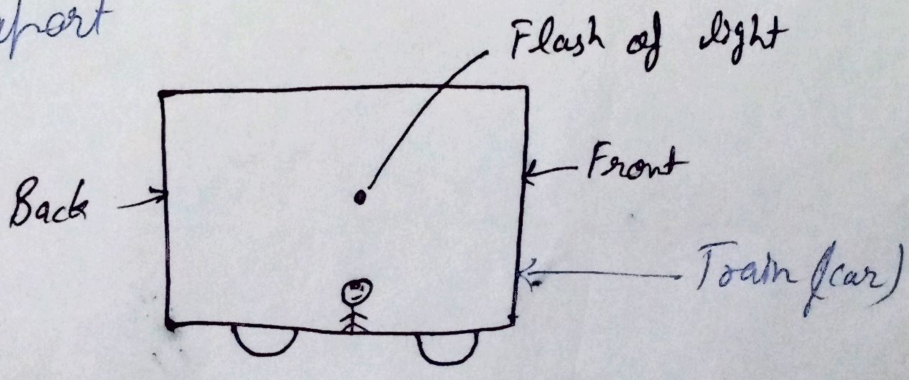
According to Einstein -

→ Simulaneity is not absolute.

"Two events which appear to take place simultaneously to one observer are, in general not simultaneous to another observer in relative motion."

Thought experiment (The train and platform)

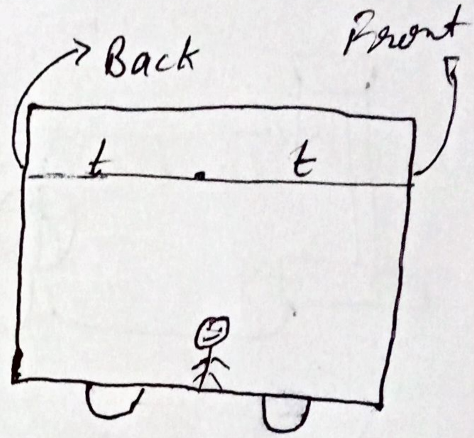
It consists of one observer midway inside a speeding traincar and another observer standing on a platform as the train moves apart



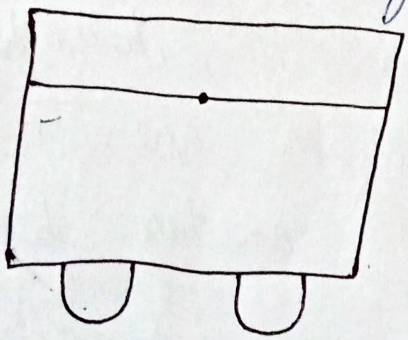
A flash of light is given off at the two obs
pass each other.

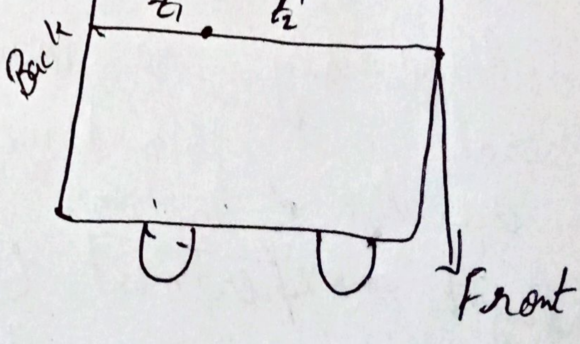
For the observer on the train

The front and back of the train can at fixed distance from the light source and as sunch according to the the observer, the light will reach the front and back of the train at the same time all observer, the light headed



For the Front. Thus, the flashes of light will strike the ends of car (train/car) at diff. times.



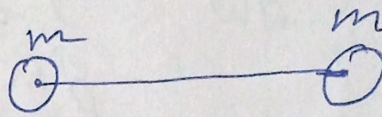


Mass - energy equivalence -

According to Einstein, the mass and energy are equivalent i.e. mass can be converted into enerny and vice-versa.

Consider an particle (such as photon) maximum moving with velocity 'v' having rest mass 'mo' and moving mass m variable with velocity then.

$$m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$$



squaring both sides;

$$m^2 = \frac{m_0^2}{1-\frac{v^2}{c^2}} = \frac{m_0^2}{\frac{c^2-v^2}{c^2}}$$

$$m^2 = \frac{m_0^2 c^2}{c^2-v^2}$$

$$m^2c^2 - m^2v^2 = m_0^2c^2$$

differentiating both sides

$$2mdm\,c^2 - [2mdm\,v^2 + 2vdv.m^2] = 0$$

$$2mdmc^2 - 2mdmv^2 - 2vdv.m^2 = 0 \quad \text{---} (1)$$

and we know that

$$P = mv$$

and, $F = \frac{\text{change in momentum}}{\text{time}}$

$$F = \frac{\Delta P}{\Delta t} \qquad \because P = mv \quad \Delta t = dt \quad \Delta P = d$$

or $F = \frac{d(mv)}{dt}$

and Now,

$$F = \frac{dm.v}{dt} + \frac{m.dv}{dt} \quad \text{——} \quad (2)$$

So, the work by moving particle to compute its energy,

$$dW = F.ds$$

$$= \left(\frac{dm\,v}{dt} + m\frac{dv}{dt}\right).ds \quad [\text{from eq}^n (2)]$$

$$= dm.v\frac{ds}{dt} + m\,dv\frac{ds}{dt}$$

$$dW = dm.v.v + m\,dv.v \qquad \left[\because \frac{ds}{dt} = v\right]$$

$$= v^2 dm + m.v\,dv \quad \text{---} \quad (3)$$

Now eq^n (1)

$$c^2 dm = v^2 dm + mv\,dv$$

from eq^n (3)

$$c^2 dm = dW$$

If $dW = dK$ then

$$c^2 dm = dK$$

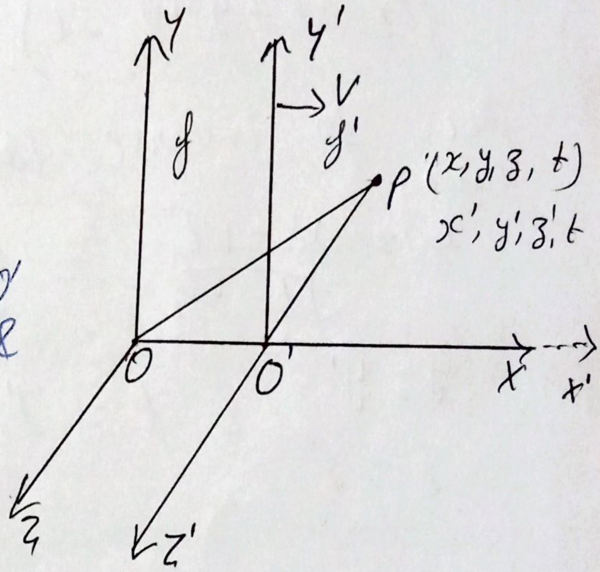
or $dK = c^2 dm$

By Integrating

$$\int_0^K dK = \int_{m_0}^{m} c^2 dm$$

Doppler's Effect -

Acc. to doppler's effect, whenever there is relative motion b/w a source of light & observer. The apparent frequency of light observed by the observer is diff. from the actual frequency of light emitted by the source of light. This effect is named after Austrian Mathematician & Physicist "Christian Doppler" who observed this effect in 1842.



Let us consider 2 inertial frames of references S & S'. Initially their origins O & O' coinside at time $t = t' = 0$. & frame S' is moving with a uniform velocity 'v' w.r.t. frame S along +x-dirⁿ.

Let the eqⁿ of plane wave in frame S is.

$$\psi = A\exp(\vec{k}\cdot\vec{r} - \omega t) \quad \text{—} (1) \quad \text{where } \omega = 2\pi f$$

$$\& \quad k = \frac{\omega}{c} \quad \frac{2\pi f}{c} = \frac{2\pi}{\frac{c}{f}} = \frac{2\pi}{\lambda}$$

$$\psi = A\exp. i(\vec{k}\cdot\vec{r} - \omega t) \quad \text{—} (1)$$

Now
$$\vec{k}\cdot\vec{r} = (k_x\hat{i} + k_y\hat{j})\cdot(x\hat{i} + y\hat{j})$$
$$= (k\cos\theta\,\hat{i} + k\sin\theta\,\hat{j})(x\hat{i} + y\hat{j})$$
$$\vec{k}\cdot\vec{r} = k(x\cos\theta + y\sin\theta)$$
$$= \frac{2\pi f}{c}(x\cos\theta + y\sin\theta) \quad \text{—} (2)$$

Using (2) in (1) we get

$$\psi = A\,exp.\,i\left[\frac{2\pi f}{c}(x\cos\theta + y\sin\theta) - 2\pi f t\right]$$

$$= A\,exp.\,i2\pi f\left(\frac{x\cos\theta + y\sin\theta}{c} - t\right) \quad \text{—(3)}$$

Similarly the eq^n of plane wave in frame S' can be written as

$$\psi = A'\,exp.\,i2\pi f'\left(\frac{x'\cos\theta' + y'\sin\theta'}{c} - t'\right) \quad \text{—(4)}$$

As we know that,

$$f\left(\frac{x\cos\theta + y\sin\theta}{c} - t\right) = f'\left(\frac{x'\cos\theta' + y'\sin\theta'}{c}\right) - t' \quad \text{—(5)}$$

Acc. to inverse Lorentz Transformation eq^s, we have

$$x = \frac{x' + vt'}{\sqrt{1 - \frac{v^2}{c^2}}} \Rightarrow x = \frac{x' + \left(\frac{v}{c}\right)ct'}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$y = y' \qquad \& \quad t = \frac{t' + \frac{v}{c^2}x'}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$z = z'$$

Let $\beta = \frac{v}{c}$

$$\gamma = \frac{1}{\sqrt{1 + v^2/c^2}} = \frac{1}{\sqrt{1 - \beta^2}} = \frac{1}{\gamma}$$

$$\left\{\left(x = \gamma(x' + \beta c t')\right);\ y = y' \ \&\ t = \gamma\left(t' + \frac{\beta x'}{c}\right)\right\} \quad \text{—(6)}$$

Using (6) in (5), we get

$$f\left[\frac{\gamma(x' + \beta c t')\cos\theta + y'\sin\theta}{c} - \gamma\left(t' + \frac{\beta}{c}x'\right)\right]$$

$$= f'\left[\frac{x'\cos\theta' + y'\sin\theta'}{c} - t'\right]$$

$$\frac{f\sin\theta}{c} = \frac{f'\sin\theta'}{c} = f\sin\theta = f'\sin\theta' \quad \text{—— (8)}$$

Dividing (8) by (7), we get

$$\frac{f\sin\theta}{f\gamma(\cos\theta-\beta)} = \frac{f'\sin\theta'}{f'\cos\theta'} = \frac{\sin\theta}{\gamma(\cos\theta-\beta)} = \tan\theta'$$

This is known as Relativistic Aberration formula.

$$\frac{f\gamma\beta c\cos\theta}{c} - f\gamma = -f' = f\gamma(\beta\cos\theta - 1) = -f''$$

$$= f\gamma(1-\beta\cos\theta) \quad \text{——}$$

$$f = \frac{f'}{\gamma(1-\beta\cos\theta)} \quad \text{—— (9)}$$

eq^n (9) describes Relativistic Doppler effect.

Longitudinal Doppler effect –

when $\theta = 0$ i.e. when source of move is moving towards the observer

$$f = \frac{f'}{\gamma(1-\beta\cos\theta)} \Rightarrow f = \frac{f'}{\gamma(1-\beta)}$$

$$f = \frac{f'\sqrt{\frac{c+v}{c}}}{\sqrt{\frac{c-v}{c}}} = \frac{f'\sqrt{c+v}}{\sqrt{c-v}}$$

$$f = \frac{f'}{(1+\frac{v}{c})^{-1/2}(1-\frac{v}{c})^{1/2}} \qquad \left\{ \begin{array}{l} \because (1+x)^n \simeq 1+nx \text{ if } x \ll 1 \\ (1-x)^n = 1-nx \text{ if } x \ll 1 \end{array} \right.$$

Non-Relativistic case i.e.

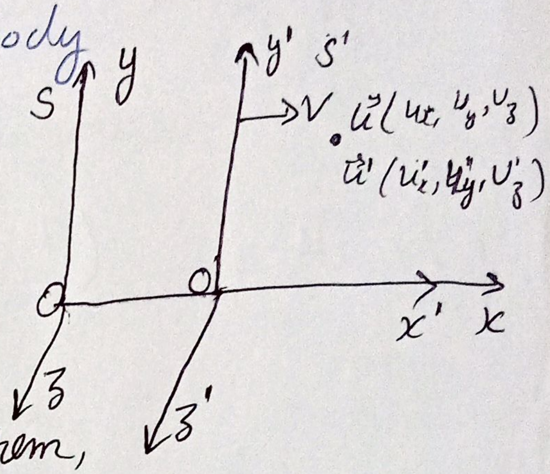
$v \ll c$ or $\frac{v}{c} \ll 1$

Transformation of energy and momentum -

Let us consider 2 ~~inter~~ inertial frame of reference S and S' such that S' frame moves with uniform velocity 'V' w.r.t. frame S which is at rest. The motion of frame S' is along +ve x-axis.

Let m & m' be the masses of a body w.r.t. frame S & S' respectively.

u & u' be the velocity of body w.r.t. frame S & S'.



$$\vec{u} = u_x \hat{i} + u_y \hat{j} + u_z \hat{k}$$

$$u^2 = u_x^2 + u_y^2 + u_z^2 \quad \text{—(1)}$$

$$u'^2 = u_x'^2 + u_y'^2 + u_z'^2 \quad \text{—(2)}$$

From velocity addition theorem, we have

$$u'_x = \frac{u_x - V}{1 - \frac{u_x V}{c^2}} \quad \text{—(3)}$$

$$u'_z = \frac{u_z\left(1 - \frac{V^2}{c^2}\right)^{1/2}}{1 - \frac{u_x V}{c^2}} \quad \text{—(5)}$$

$$u'_y = \frac{u_y\left(1 - \frac{V^2}{c^2}\right)^{1/2}}{1 - \frac{u_x V}{c^2}} \quad \text{—(4)}$$

Using (3), (4) & (5) in (2), we get

$$u^2 = \left[\frac{u_x - V}{1 - \frac{u_x V}{c^2}}\right] + \left[\frac{u_y\left(1 - \frac{V^2}{c^2}\right)^{1/2}}{1 - \frac{u_x V}{c^2}}\right] + \left[\frac{u_z\left(1 - \frac{V^2}{c^2}\right)^{1/2}}{1 - \frac{u_x V}{c^2}}\right]^2$$

$$= \frac{1}{\left(1 - \frac{u_x V}{c^2}\right)^2}\left[(u_x - V)^2 + u_y^2\left(1 - \frac{V^2}{c^2}\right) + u_z^2\left(1 - \frac{V^2}{c^2}\right)\right]$$

$$= \frac{1}{\left(1 - \frac{u_x V}{c^2}\right)^2}\left[u^2 + V^2 - 2u_x V - u_y^2\frac{V^2}{c^2} - u_z^2\frac{V^2}{c^2}\right]$$

Dividing both sides by 'c^2' we get

$$\frac{u'^2}{c^2} = \frac{1}{c^2\left(1-\frac{u_x v}{c^2}\right)^2}\left[u^2+v^2-2u_x v - u_y^2\frac{v^2}{c^2} - u_z^2\frac{v^2}{c^2}\right]$$

Subtracting L.H.S & R.H.S from 1 we get

$$1-\frac{u'^2}{c^2} = 1 - \frac{1}{c^2\left[1-\frac{u_x v}{c^2}\right]^2}\left[u^2+v^2-2u_x v - u_y^2\frac{v^2}{c^2} - u_z^2\frac{v^2}{c^2}\right]$$

$$1-\frac{u'^2}{c^2} = \frac{\left(c^2+\frac{c^2u_x^2v^2}{c^4} - \frac{c^2 2u_x v}{c^2}\right) - u^2 - v^2 + 2u_x v + u_y^2\frac{v^2}{c^2} + u_z^2\frac{v^2}{c^2}}{c^2\left(1-\frac{u_x v}{c^2}\right)^2}$$

$$1-\frac{u'^2}{c^2} = \frac{\left(1-\frac{u^2}{c^2}\right)c^2\left(1-\frac{v^2}{c^2}\right)}{c^2\left(1-\frac{u_x v}{c^2}\right)^2}$$

$$1-\frac{u'^2}{c^2} = \frac{\left(1-\frac{u^2}{c^2}\right)\left(1-\frac{v^2}{c^2}\right)}{\left(1-\frac{u_x v}{c^2}\right)^2} \text{ ——— } (6)$$

Now $m = \frac{m_0}{\sqrt{1-\frac{u^2}{c^2}}} \longrightarrow (7)$, $m' = \frac{m_0}{\sqrt{1-\frac{u'^2}{c^2}}} \text{ —— } (8)$

Dividing (7) by (8) we get

$$\frac{m}{m'} = \frac{\frac{m_0}{\sqrt{1-\frac{u^2}{c^2}}}}{\frac{m_0}{\sqrt{1-\frac{u'^2}{c^2}}}} = \frac{\left(1-\frac{u'^2}{c^2}\right)^{1/2}}{\left(1-\frac{u^2}{c^2}\right)^{1/2}} \longleftarrow (9)$$

Using (6) in R.H.S of eqn (9), we get

$$\frac{m}{m'} = \left[\frac{\left(1-\frac{u^2}{c^2}\right)\left(1-\frac{v^2}{c^2}\right)}{\left(1-\frac{u_x v}{c^2}\right)^2}\right]^{1/2} \cdot \frac{1}{\left(1-\frac{u^2}{c^2}\right)^{1/2}}$$

$$\frac{m}{m_1} = \frac{\left(1-\frac{v^2}{c^2}\right)^{1/2}}{\left(1-\frac{u_x v}{c^2}\right)}$$

$$m'\left(1-\frac{v^2}{c^2}\right)^{1/2} = m\left(1-\frac{u_x v}{c^2}\right)$$

$$m' = \frac{m\left(1-\frac{u_x v}{c^2}\right)}{\left(1-\frac{v^2}{c^2}\right)^{1/2}} = \gamma m\left(1-\frac{u_x v}{c^2}\right) \text{ --- } (10)$$

where $\gamma = \frac{1}{\left(1-\frac{v^2}{c^2}\right)^{1/2}}$

Momentum in frame S & S' respectively

$P_x = mu_x$; $P_y = mu_y$, $P_z = mu_z$ ——

and $P_x' = m'u_x'$ —— (a) $P_y' = m'u_y'$ —— (b)

$P_z' = m'u_z'$ —— (c)

Using (10) & (3) in (a) we get

$$P_x' = \left[\gamma m\left(1-\frac{u_x v}{c^2}\right)\right]\left[\frac{u_x - v}{1-\frac{u_x v}{c^2}}\right] = \gamma(mu_x - mv)$$

$$P_x' = \gamma\left[P_x - \frac{Ev}{c^2}\right) \text{ --- } (11) \qquad \left[\because E = mc^2,\ m = \frac{E}{c^2}\right]$$

Using (10) & (4) in (b) we get

$$P_y' = \left[\gamma m\left(1-\frac{u_x v}{c^2}\right)\right]\left[\frac{u_y\left(1-\frac{v^2}{c^2}\right)^{1/2}}{1-\frac{u_x v}{c^2}}\right]$$

$$P_y' = P_y \text{ --- } (12)$$

Similarly $P_z' = P_z$ —— (13)

Thus complete set of Lorentz transformation eq^n for momentum & energy are

$$P_x' = \gamma\left(P_x - \frac{E}{c^2}v\right) \text{ --- }$$

$P_y' = P_y$ and $P_z' = P_z$

Using (16) in (13) we get

$$E' = \left[\gamma m\left(1 - \frac{u_x v}{c^2}\right)\right]c^2$$

$$= \gamma\left[mc^2 - mc^2\frac{u_x v}{c^2}\right] \quad [\because p_x = mu_x]$$

$$\boxed{E' = \gamma\left[E - p_x v\right]} \quad \text{—— (17)}$$

☆ Energy momentum for 4-vector –

The momentum 4-vector is denoted as p_μ & it has 4 components p_1, p_2, p_3 & p_4 in 4-D Minkowski space. i.e. $p_\mu = (p_1, p_2, p_3, p_4)$ —— (1)

Now $p_1 = m_0 u_1$ where m_0 = rest mass of particle

& $u_1 = \frac{u_x}{\sqrt{1 - u^2/c^2}}$ = Ist component of velocity 4-vector

$$\Rightarrow p_1 = m_0 \frac{u_x}{\sqrt{1 - u^2/c^2}} = \left(\frac{m_0}{\sqrt{1 - \frac{u^2}{c^2}}}\right) u_x \quad \text{—— (2)}$$

Now from variation of mass with velocity formula.

$$m = \frac{m_0}{\sqrt{1 - \frac{u^2}{c^2}}} \quad \text{—— (3)}$$

Using (3) in (2) we get

$$p_1 = mu_x$$

$$\boxed{p_1 = p_x} \quad \text{—— (4)}$$

where $p_x = mu_x$ = x-component of momentum vector in 3D space

Similarly,

$p_2 = p_y$ & $p_3 = p_z$ —— (5)

Now $p_4 = m_0 U_4$ —— (6)

Using (3) in (6) we get

$p_4 = m\, ic$ —— (7)

Now $E = mc^2 \Rightarrow m = \frac{E}{c^2}$ —— (8)

Using (8) in (7) we get

$p_4 = \left(\frac{E}{c^2}\right) ic \Rightarrow p_4 = \frac{iE}{c}$ —— (9)

Using (4), (5) & (9) in (1), we get

$p_\mu = \left(p_x, p_y, p_z, \frac{iE}{c}\right)$

or $p_\mu = \left(\vec{p}, \frac{iE}{c}\right)$

where $\vec{p} = p_x \hat{i} + p_y \hat{j} + p_z \hat{k}$

Momentum vector in 3-D space

$p = \sqrt{p_x^2 + p_y^2 + p_z^2}$

$p^2 = p_x^2 + p_y^2 + p_z^2$ —— (10)

Now square of magnitude of momentum 4-vector is given as

$p_\mu^2 = p_1^2 + p_2^2 + p_3^2 + p_4^2$

$p_\mu p_\mu = p_x^2 + p_y^2 + p_z^2 + \left(\frac{iE}{c}\right)^2$ —— (11)

Using (10) in (11) we get

$p_\mu p_\mu = p^2 + i^2 \frac{E^2}{c^2}$

$= p^2 - \frac{E^2}{c^2}$

$$p_\mu p_\mu = \frac{p^2c^2 - E^2}{c^2}$$

$$p_\mu p_\mu = \frac{-(E^2 - p^2c^2)}{c^2} \quad \text{—— (12)}$$

Acc. to relativistic energy - momentum relationship we have

$$E^2 = p^2c^2 + m_0^2c^4$$

$$E^2 - p^2c^2 = m_0^2c^4 \quad \text{—— (13)}$$

Using (13) in (12) we get

$$p_\mu p_\mu = \frac{-(m_0^2c^4)}{c^2} = -m_0^2c^2$$

$$\boxed{p_\mu p_\mu = -m_0^2c^2} \quad \text{—— (14)}$$